Regular Reduction of Controlled Hamiltonian System with Symplectic Structure and Symmetry

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Abstract: In this paper, our goal is to study the regular reduction theory of regular controlled Hamiltonian (RCH) systems with symplectic structure and symmetry, and this reduction is an extension of regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. Thus, in order to describe uniformly RCH systems defined on a cotangent bundle and on its regular reduced spaces, we first define a kind of RCH systems on a symplectic fiber bundle. Then introduce regular point and regular orbit reducible RCH systems with symmetry by using momentum map and the associated reduced symplectic forms. Moreover, we give regular point and regular orbit reduction theorems for RCH systems to explain the relationships between RpCH-equivalence, RoCH-equivalence for reducible RCH systems with symmetry and RCH-equivalence for associated reduced RCH systems. Finally, as an application we regard rigid body and heavy top as well as them with internal rotors as the regular point reducible RCH systems on the rotation group SO(3) and on the Euclidean group SE(3), as well as on their generalizations, respectively, and discuss their RCH-equivalence. We also describe the RCH system and RCH-equivalence from the viewpoint of port Hamiltonian system with a symplectic structure.

Keywords: regular controlled Hamiltonian system, symplectic structure, momentum map, regular Hamiltonian reduction, RCH-equivalence.

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1 Introduction

Symmetry is a general phenomenon in the natural world, but it is widely used in the study of mathematics and mechanics. The reduction theory for mechanical system with symmetry has its origin in the classical work of Euler, Lagrange, Hamilton, Jacobi, Routh, Liouville and Poincaré and its modern geometric formulation in the general context of symplectic manifolds and equivariant momentum maps is developed by Meyer, Marsden and Weinstein; see Abraham and Marsden [1] or Marsden and Weinstein [23] and Meyer [24]. The main goal of reduction theory in mechanics is to use conservation laws and the associated symmetries to reduce the number of dimensions of a mechanical system required to be described. So, such reduction theory is regarded as a useful tool for simplifying and studying concrete mechanical systems. Reduction is a very general procedure that is applied to arbitrary dynamical systems with symmetry. However, it is particularly powerful for conservative systems whose symmetries are induced by a momentum map; see Abraham and Marsden [1], Arnold [3], Marsden [20], Marsden et al [21], Marsden and Ratiu [22] and Ortega and Ratiu [26] for more details.

It is well-known that Hamiltonian reduction theory is one of the most active subjects in the study of modern analytical mechanics and applied mathematics, in which a lot of deep and beautiful results have been obtained, see the studies given by Abraham and Marsden [1], Arnold [3], Leonard and Marsden [19], Marsden et al [20–23], Ortega and Ratiu [26] etc. on regular point reduction and regular orbit reduction, singular point reduction and singular orbit reduction, optimal reduction and reduction by stages for Hamiltonian systems and so on; and there is still much to be done in this subject.

On the other hand, just as we have known that the theory of mechanical control systems presents a challenging and promising research area between the study of classical mechanics and modern nonlinear geometric control theory and there have been a lot of interesting results. Such as, Bloch et al. in [5–8], referred to the use of feedback control to realize a modification to the structure of a given mechanical system; Blankenstein et al. in [4], Crouch and Van der Schaft in [12], Nijmeijer and Van der Schaft in [25], van der Schaft in [27–31], referred to the reduction and control of implicit (port) Hamiltonian systems, and to the use of feedback control to stabilize mechanical systems and so on.

Nevertheless, we also note that Chang et al. in [9], defined a controlled Hamiltonian (CH) system by using almost Poisson tensor, and studied the reduction of CH systems with symmetry in [11]. But, because for the CH systems and their reduced CH systems defined in [9, 11], the authors have not given directly the spaces on which these systems are defined, see Definition 3.1 in [9] and Definition 3.1, 3.3 in [11]. Thus, it is impossible to state clearly the CH-equivalence and reduced CH-equivalence, by comparing the Definition 3.2, Definition 4.3 and Definition 5.3 in our paper with the Definition 3.6 and Definition 3.8 in [11]. Moreover, because the authors do not use the momentum map in their Hamiltonian reduction of CH system, it is also impossible to determine precisely the reduced spaces of CH systems, and it is not that all of CH systems in [11] have same space T^*Q, same action of Lie group G, and same reduced space T^*Q/G. For example, we consider the cotangent bundle T^*Q of a smooth manifold Q with a free and proper action of Lie group G, and the Poisson tensor B on T^*Q is determined by canonical symplectic form ω_0 on T^*Q. Then there is an Ad^*-equivariant momentum map $\mathbf{J} : T^*Q \rightarrow \mathfrak{g}^*$ for the symplectic, free and proper cotangent lifted G-action, where $\mathfrak{g}^*$ is the dual of Lie algebra $\mathfrak{g}$ of G. For $\mu \in \mathfrak{g}^*$, a regular value of $\mathbf{J} : M \rightarrow \mathfrak{g}^*$, from Abraham and Marsden [1], we know that regular point reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ and regular orbit reduced space $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ at μ are not T^*Q/G, and the two reduced spaces are determined by the momentum map $\mathbf{J}$, where G_μ is the isotropy subgroup of coadjoint G-action at the point μ, and $\mathcal{O}_\mu$ is the orbit of coadjoint G-action through the point μ. Thus, in the two cases, it is impossible to determine the reduced spaces of CH systems by using the method given in Chang et al [11].

In order to deal with the above problems and determine the reduced CH systems, our idea in this paper is that we first define a CH system on T^*Q by using a symplectic form, and such system is called a RCH system, and then regard the associated Hamiltonian system on T^*Q as a spacial case of the RCH system without external force and control. Thus, the set of Hamiltonian systems on T^*Q is a subset of the set of RCH systems on T^*Q. We hope to study regular reduction theory of RCH systems with symplectic structure and symmetry, as an extension of regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. The main contributions in this paper is given as follows. (1) In order to describe uniformly RCH systems defined on a cotangent bundle and on its regular reduced spaces, we define a kind of RCH systems on a symplectic fiber bundle by using its symplectic form; (2) We give regular point and regular orbit reducible RCH systems by using momentum map and the associated reduced symplectic forms, and prove regular point and regular orbit reduction theorems for RCH systems (see Theorem 4.4 and 5.4); (3) We prove that rigid body with external force torque, rigid body with internal rotors and heavy top with internal rotors are all RCH systems, and as a pair of regular point reduced RCH systems, rigid body with internal rotors (or external force torque) and heavy top with internal rotors are RCH-equivalent; (4) We describe the RCH system from the viewpoint of port Hamiltonian system with a symplectic structure, and state the relationship between RCH-equivalence of RCH system and equivalence of port Hamiltonian system.

A brief of outline of this paper is as follows. In the second section, we review some relevant definitions and basic facts about momentum map, symplectic fiber bundle, Lie group lifted actions on (co-)tangent bundles and reduction, which will be used in subsequent sections. The RCH systems are defined by using the symplectic forms on a symplectic fiber bundle and on the cotangent bundle of a configuration manifold, respectively, and RCH-equivalence is introduced in the third section. From the fourth section we begin to discuss the RCH systems with symmetry by combining with regular symplectic reduction theory. The regular point and regular orbit reducible RCH systems are considered respectively in the fourth section and the fifth section, and give the regular point and regular orbit reduction theorems for RCH systems to explain the relationships between the RpCH-equivalence, RoCH-equivalence for reducible RCH systems with symmetry and the RCH-equivalence for associated reduced RCH systems. As the applications of the theoretical results, in sixth section, we first give the regular point reduced RCH systems on a Lie group G and on its generalization $G \times V$, which are the RCH systems on a coadjoint orbit $\mathcal{O}_\mu$ of G and on its generalization $\mathcal{O}_\mu \times V \times V^*$. Then we describe uniformly the rigid body and heavy top as well as them with internal rotors as the regular point reducible RCH systems on the rotation group SO(3) and on the Euclidean group SE(3), as well as on their generalizations, respectively, and give their regular point reduced RCH systems and discuss their RCH-equivalence. In order to understand well the abstract definition of RCH system, we also describe the RCH system and RCH-equivalence from the viewpoint of port Hamiltonian system with a symplectic structure. These research work develop the theory of Hamiltonian reduction for the regular controlled Hamiltonian systems with symmetry and make us have much deeper understanding and recognition for the structure of controlled Hamiltonian systems.

2 Preliminaries

In order to study the regular reduction theory of RCH systems, we first give some relevant definitions and basic facts about momentum maps, symplectic fiber bundle, Lie group lifted actions on (co-)tangent bundles and reduction, which will be used in subsequent sections, we shall follow the notations and conventions introduced in Abraham et al [1, 2], Marsden [20], Marsden et al [21], Marsden and Ratiu [22], Ortega and Ratiu [26], Kobayashi and Nomizu [16]. In this paper, we assume that all manifolds are real, smooth and finite dimensional and all actions are smooth left actions.

2.1 Momentum map

Let (M,ω) be a symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$. We say that G acts on M and the action of any $g \in G$ on $z \in M$ will be denoted by $\Phi : G \times M \to M : \Phi(g,z) = g \cdot z$. For any $g \in G$, the map $\Phi_g := \Phi(g,\cdot) : M \to M$ is a diffeomorphism of M and if the map Φ_g satisfies $\Phi_g^*\omega = \omega$, $\forall g \in G$, we say that G acts symplectically on a symplectic manifold (M,ω). The isotropy subgroup of a point $z \in M$ is $G_z = \{g \in G|\ g \cdot z = z\}$. An action is free if all the isotropy subgroups G_z are trivial; and is proper if the map $(g,z) \to (g, g\cdot z)$ is proper (i.e., the pre-image of every compact set is compact). For a proper action, all isotropy subgroups are compact. The G-orbit of $z \in M$ is denoted by $\mathcal{O}_z = G \cdot z = \{\Phi_g(z)|\ g \in G\}$, and the orbit space by $M/G = \{\mathcal{O}_z|\ z \in M\}$. If G acts freely and properly on M, then M/G has a unique smooth structure such that $\pi_G : M \to M/G$ is a surjective submersion. If G acts only properly on M, does not act freely, then M/G is not necessarily smooth manifold, but just a quotient topological space.

For each $\xi \in \mathfrak{g}$, the infinitesimal generator of ξ is the vector field ξ_M defined by $\xi_M(z) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(t\xi)\cdot z, \forall z \in M$. We will also write $\xi_M(z)$ as $\xi\cdot z$, and refer to the map $(\xi, z) \mapsto \xi\cdot z$ as the infinitesimal action of $\mathfrak{g}$ on M. A momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ is defined by $< \mathbf{J}(z), \xi >= J_\xi(z)$, for every $\xi \in \mathfrak{g}$, where the function $J_\xi : M \to \mathbb{R}$ satisfies $X_{J_\xi} = \xi_M$, and $\mathfrak{g}^*$ is the dual of Lie algebra $\mathfrak{g}$, and $<,>: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the duality pairing between the dual $\mathfrak{g}^*$ and $\mathfrak{g}$. If the adjoint action of G on $\mathfrak{g}$ is denoted by Ad, and the infinitesimal adjoint action by ad, then the coadjoint action of G on g^* is the inverse dual to the adjoint action, given by $g\cdot\nu = \mathrm{Ad}^*_{g^{-1}}\nu = (\mathrm{Ad}_{g^{-1}})^*\nu, \forall\ \nu \in \mathfrak{g}^*$. The infinitesimal coadjoint action is given by $\xi \cdot \nu = -\,\mathrm{ad}^*_\xi\,\nu, \forall\ \nu \in \mathfrak{g}^*$. For $\mu \in \mathfrak{g}^*$, a value of $\mathbf{J} : M \to \mathfrak{g}^*$, G_μ denotes the isotropy subgroup of G with respect to the coadjoint G-action

$\mathrm{Ad}^*_{g^{-1}}$ at the point μ, and $\mathcal{O}_\mu$ denotes the G-orbit of through the point μ in $\mathfrak{g}^*$. The momentum map $\mathbf{J}$ is Ad^*-equivariant if $\mathbf{J}(\Phi_g(z)) = \mathrm{Ad}^*_{g^{-1}}\mathbf{J}(z)$, for any $z \in M$.

The following proposition is very important for the regular reduction and singular reduction of Hamiltonian systems with symmetry; see Marsden [20] and Ortega and Ratiu [26].

Proposition 2.1 *(Bifurcation Lemma) Let (M, ω) be a symplectic manifold and G a Lie group acting symplectically on M (not necessarily freely). Suppose that the action has an associated momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Then for any $z \in M$, $(\mathfrak{g}_z)^0 = range(T_z\mathbf{J})$, where $\mathfrak{g}_z = \{\xi \in \mathfrak{g}|\ \xi_M(z) = 0\}$ is the Lie algebra of the isotropy subgroup $G_z = \{g \in G|\ g \cdot z = z\}$ and $(\mathfrak{g}_z)^0 = \{\mu \in \mathfrak{g}^*|\ \mu|_{\mathfrak{g}_z} = 0\}$ denotes the annihilator of $\mathfrak{g}_z$ in $\mathfrak{g}^*$.*

An immediate consequence of this proposition is the fact that when the action of G is free, each value $\mu \in \mathfrak{g}^*$ of the momentum map $\mathbf{J}$ is regular. Thus, if μ is a singular value of $\mathbf{J}$, then the G-action is not free. In addition, if μ is a regular value of $\mathbf{J}$ and $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$, the $\mathbf{J}$ is transverse to $\mathcal{O}_\mu$ and hence $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is automatically an embedded submanifold of M. In this paper, we consider only that the G-action is free, and the Hamiltonian reductions are regular.

2.2 Symplectic fiber bundles

Let E and M be two smooth manifolds, Lie group G acts freely on E from the left side. Denote by (E, M, π, G) a (left) principal fiber bundle over M with group G, where E is the bundle space, M is the base space, G is the structure group and the projection $\pi : E \to M$ is a surjective submersion. For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of E, which is called the fiber over x. Each fiber of the principal bundle (E, M, π, G) is diffeomorphic to G. In the following we shall give a construction of the associated bundle of G-principal bundle. Assume that F is another smooth manifold and Lie group G acts on F from the left side. We can define a fiber bundle associated to principal bundle (E, M, π, G) with fiber F as follows. Consider the left action of G on the product manifold $E \times F$, $\Phi : G \times (E \times F) \to E \times F$ given by $\Phi(g, (z, y)) = (gz, g^{-1}y),\ \forall\ g \in G,\ z \in E,\ y \in F$. Denote by $E \times_G F$ is the orbit space $(E \times F)/G$, and the map $\rho : E \times_G F \to M$ is uniquely determined by the condition $\rho \cdot \pi_{/G} = \pi \cdot \pi_E$, that is, the following commutative Diagram-1,

$$\begin{array}{ccc} E \times F & \xrightarrow{\pi_{/G}} & E \times_G F \\ {\scriptstyle \pi_E}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho} \\ E & \xrightarrow{\pi} & M \end{array}$$

Diagram-1

where $\pi_{/G} : E \times F \to E \times_G F$ is the canonical projection and $\pi_E : E \times F \to E$ is the projection onto the first factor. Then $(E \times_G F, M, F, \rho, G)$, simply written as (E, M, F, π, G), is a fiber bundle with fiber F and structure group G associated to principal bundle (E, M, π, G). In particular, if $F = V$ is a vector space, then (E, M, V, π, G) is a vector bundle associated to principal bundle (E, M, π, G).

A bundle of symplectic manifolds is such a fiber bundle (E, M, F, π, G), all of whose fibers are symplectic and whose structure group G preserves the symplectic structure on F. From Gotay et al. [14] we know that there exists a presymplectic form ω_E on E under some topological conditions, whose pull-back to each fiber is the given fiber symplectic form. We assume that if a symplectic form ω_E is given on E, then (E, ω_E) is called a symplectic fiber bundle. In particular, if E is a vector bundle, then (E, ω_E) is called a symplectic vector bundle; see Libermann and Marle [18].

2.3 Lie group lifted action on (co-)tangent bundles and reduction

For a smooth manifold Q, its cotangent bundle T^*Q has a canonical symplectic form ω_0, which is given in natural cotangent bundle coordinates (q^i, p_i) by $\omega_0 = \mathbf{d}q^i \wedge \mathbf{d}p_i$, so T^*Q is a symplectic vector bundle. Let $\Phi : G \times Q \to Q$ be a left smooth action of a Lie group G on the manifold Q. The tangent lift of this action $\Phi : G \times Q \to Q$ is the action of G on TQ, $\Phi^T : G \times TQ \to TQ$ given by $g \cdot v_q = T\Phi_g(v_q),\ \forall\ v_q \in T_qQ, q \in Q$. The cotangent lift is the action of G on T^*Q, $\Phi^{T^*} : G \times T^*Q \to T^*Q$ given by $g \cdot \alpha_q = (T\Phi_{g^{-1}})^* \cdot \alpha_q,\ \forall\ \alpha_q \in T^*_qQ,\ q \in Q$. The tangent or cotangent lift of any proper (resp. free) G-action is proper (resp. free). Each cotangent lift action is symplectic with respect to the canonical symplectic form ω_0, and has an Ad^*-equivariant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ given by $< \mathbf{J}(\alpha_q), \xi >= \alpha_q(\xi_Q(q))$, where $\xi \in \mathfrak{g}$, $\xi_Q(q)$ is the value of the infinitesimal generator ξ_Q of the G-action at $q \in Q$, $<,>: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the duality pairing between the dual $\mathfrak{g}^*$ and $\mathfrak{g}$.

The reduction theory of cotangent bundle is a very important special case of general reduction theory. Let $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $\mathbf{J}$, the simplest case of symplectic reduction of cotangent bundle T^*Q is regular point reduction at zero, in this case the symplectic reduced space formed at $\mu = 0$ is given by $((T^*Q)_\mu, \omega_\mu) = (T^*(Q/G), \omega_0)$, where ω_0 is the canonical symplectic form of cotangent bundle $T^*(Q/G)$. Thus, the reduced space $((T^*Q)_\mu, \omega_\mu)$ at $\mu = 0$ is a symplectic vector bundle. If $\mu \neq 0$, from Marsden et al [21] we know that, when $G_\mu = G$, the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to symplectic vector bundle $(T^*(Q/G), \omega_0 - B_\mu)$, where B_μ is a magnetic term; If G is not Abelian and $G_\mu \neq G$, the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to a symplectic fiber bundle over $T^*(Q/G_\mu)$ with fiber to be the coadjoint orbit $\mathcal{O}_\mu$. In the case of regular orbit reduction, from Ortega and Ratiu [26] and the regular reduction diagram, we know that the regular orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$, and hence is symplectically diffeomorphic to a symplectic fiber bundle. To sum up above discussion, if we may define a RCH system on a symplectic fiber bundle, then it is possible to describe uniformly the RCH systems on T^*Q and their regular reduced RCH systems on the associated reduced spaces.

3 Regular Controlled Hamiltonian Systems

In this paper, our goal is to study regular reduction theory of RCH systems with symplectic structure and symmetry, as an extension of regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. Thus, in order to describe uniformly RCH systems defined on a cotangent bundle and on its regular reduced spaces, in this section we first define a RCH system on a symplectic fiber bundle. In particular, we obtain the RCH system by using the symplectic structure on the cotangent bundle of a configuration manifold as a special case, and discuss RCH-equivalence. In consequence, we can study the RCH systems with symmetry by combining with regular symplectic reduction theory of Hamiltonian systems. For convenience, we assume that all controls appearing in this paper are the admissible controls.

Let (E, M, N, π, G) be a fiber bundle and (E, ω_E) be a symplectic fiber bundle. If for any function $H : E \to \mathbb{R}$, we have a Hamiltonian vector field X_H by $i_{X_H}\omega_E = \mathbf{d}H$, then (E, ω_E, H) is a Hamiltonian system. Moreover, if considering the external force and control, we can define a kind of regular controlled Hamiltonian (RCH) system on the symplectic fiber bundle E as follows.

Definition 3.1 *(RCH System) A RCH system on E is a 5-tuple (E, ω_E, H, F, W), where (E, ω_E, H) is a Hamiltonian system, and the function $H : E \to \mathbb{R}$ is called the Hamiltonian, a fiber-preserving map $F : E \to E$ is called the (external) force map, and a fiber submanifold W of E is called the control subset.*

Sometimes, W also denotes the set of fiber-preserving maps from E to W. When a feedback control law $u : E \to W$ is chosen, the 5-tuple (E, ω_E, H, F, u) denotes a closed-loop dynamic system. In particular, when Q is a smooth manifold, and T^*Q its cotangent bundle with a symplectic form ω (not necessarily canonical symplectic form), then (T^*Q, ω) is a symplectic vector bundle. If we take that $E = T^*Q$, from above definition we can obtain a RCH system on the cotangent bundle T^*Q, that is, 5-tuple (T^*Q, ω, H, F, W). Where the fiber-preserving map $F : T^*Q \to T^*Q$ is the (external) force map, that is the reason that the fiber-preserving map $F : E \to E$ is called an (external) force map in above definition.

In order to describe the dynamics of the RCH system (E, ω_E, H, F, W) with a control law u, we need to give a good expression of the dynamical vector field of RCH system. At first, we introduce a notations of vertical lift maps of a vector along a fiber. For a smooth manifold E, its tangent bundle TE is a vector bundle, and for the fiber bundle $\pi : E \to M$, we consider the tangent mapping $T\pi : TE \to TM$ and its kernel $ker(T\pi) = \{\rho \in TE | T\pi(\rho) = 0\}$, which is a vector subbundle of TE. Denote by $VE := ker(T\pi)$, which is called a vertical bundle of E. Assume that there is a metric on E, and we take a Levi-Civita connection $\mathcal{A}$ on TE, and denote by $HE := ker(\mathcal{A})$, which is called a horizontal bundle of E, such that $TE = HE \oplus VE$. For any $x \in M$, $a_x, b_x \in E_x$, any tangent vector $\rho(b_x) \in T_{b_x}E$ can be split into horizontal and vertical parts, that is, $\rho(b_x) = \rho^h(b_x) \oplus \rho^v(b_x)$, where $\rho^h(b_x) \in H_{b_x}E$ and $\rho^v(b_x) \in V_{b_x}E$. Let γ be a geodesic in E_x connecting a_x and b_x, and denote by $\rho^v_\gamma(a_x)$ a tangent vector at a_x, which is a parallel displacement of the vertical vector $\rho^v(b_x)$ along the geodesic γ from b_x to a_x. Since the angle between two vectors is invariant under a parallel displacement along a geodesic, then $T\pi(\rho^v_\gamma(a_x)) = 0$, and hence $\rho^v_\gamma(a_x) \in V_{a_x}E$. Now, for $a_x, b_x \in E_x$ and tangent vector $\rho(b_x) \in T_{b_x}E$, we can define the vertical lift map of a vector along a fiber given by

$$\mathrm{vlift} : TE_x \times E_x \to TE_x; \;\; \mathrm{vlift}(\rho(b_x), a_x) = \rho^v_\gamma(a_x).$$

It is easy to check from the basic fact in differential geometry that this map does not depend on the choice of γ. If $F : E \to E$ is a fiber-preserving map, for any $x \in M$, we have that $F_x : E_x \to E_x$ and $TF_x : TE_x \to TE_x$, then for any $a_x \in E_x$ and $\rho \in TE_x$, the vertical lift of ρ under the action of F along a fiber is defined by

$$(\mathrm{vlift}(F_x)\rho)(a_x) = \mathrm{vlift}((TF_x\rho)(F_x(a_x)), a_x) = (TF_x\rho)^v_\gamma(a_x),$$

where γ is a geodesic in E_x connecting $F_x(a_x)$ and a_x.

In particular, when $\pi : E \to M$ is a vector bundle, for any $x \in M$, the fiber $E_x = \pi^{-1}(x)$ is a vector space. In this case, we can choose the geodesic γ to be a straight line, and the vertical vector is invariant under a parallel displacement along a straight line, that is, $\rho^v_\gamma(a_x) = \rho^v(b_x)$. Moreover, when $E = T^*Q$, by using the local trivialization of TT^*Q, we have that $TT^*Q \cong TQ \times T^*Q$. Because of $\pi : T^*Q \to Q$, and $T\pi : TT^*Q \to TQ$, then in this case, for any $\alpha_x,\ \beta_x \in T^*_xQ,\ x \in Q$, we know that $(0, \beta_x) \in V_{\beta_x}T^*_xQ$, and hence we can get that

$$\mathrm{vlift}((0, \beta_x)(\beta_x), \alpha_x) = (0, \beta_x)(\alpha_x) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (\alpha_x + s\beta_x),$$

which is consistent with the definition of vertical lift map along fiber in Marsden and Ratiu [22].

For a given RCH System (T^*Q, ω, H, F, W), the dynamical vector field of the associated Hamiltonian system (T^*Q, ω, H) is that $X_H = (\mathbf{d}H)^\sharp$, where, $\sharp : T^*T^*Q \to TT^*Q; \mathbf{d}H \mapsto (\mathbf{d}H)^\sharp$, such that $i_{(\mathbf{d}H)^\sharp}\omega = \mathbf{d}H$. If considering the external force $F : T^*Q \to T^*Q$, by using the above notations of vertical lift maps of a vector along a fiber, the change of X_H under the action of F is that

$$\mathrm{vlift}(F)X_H(\alpha_x) = \mathrm{vlift}((TFX_H)(F(\alpha_x)), \alpha_x) = (TFX_H)^v_\gamma(\alpha_x),$$

where $\alpha_x \in T^*_xQ,\ x \in Q$ and γ is a straight line in T^*_xQ connecting $F_x(\alpha_x)$ and α_x. In the same way, when a feedback control law $u : T^*Q \to W$ is chosen, the change of X_H under the action of u is that

$$\mathrm{vlift}(u)X_H(\alpha_x) = \mathrm{vlift}((TuX_H)(u(\alpha_x)), \alpha_x) = (TuX_H)^v_\gamma(\alpha_x).$$

In consequence, the dynamical vector field of a RCH system (T^*Q, ω, H, F, W) with a control law u is the synthetic of Hamiltonian vector field X_H and its changes under the actions of the external force F and control u, that is,

$$X_{(T^*Q,\omega,H,F,u)}(\alpha_x) = X_H(\alpha_x) + \mathrm{vlift}(F)X_H(\alpha_x) + \mathrm{vlift}(u)X_H(\alpha_x),$$

for any $\alpha_x \in T^*_xQ,\ x \in Q$. For convenience, it is simply written as

$$X_{(T^*Q,\omega,H,F,u)} = (\mathbf{d}H)^\sharp + \mathrm{vlift}(F) + \mathrm{vlift}(u). \tag{3.1}$$

We also denote that $\mathrm{vlift}(W) = \bigcup\{\mathrm{vlift}(u)X_H |\ u \in W\}$. For the RCH system (E, ω_E, H, F, W) with a control law u, we have also a similar expression of its dynamical vector field. It is worthy of note that in order to deduce and calculate easily, we always use the simple expression of dynamical vector field $X_{(T^*Q,\omega,H,F,u)}$. Moreover, we also use the simple expressions for R_P-reduced vector field $X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)}$ and R_O-reduced vector field $X_{((T^*Q)_{\mathcal{O}_\mu},\omega_{\mathcal{O}_\mu},h_{\mathcal{O}_\mu},f_{\mathcal{O}_\mu},u_{\mathcal{O}_\mu})}$ in §4 and §5.

Next, we note that when a RCH system is given, the force map F is determined, but the feedback control law $u : T^*Q \to W$ could be chosen. In order to describe the feedback control law to modify the structure of RCH system, the Hamiltonian matching conditions and RCH-equivalence are induced as follows.

Definition 3.2 *(RCH-equivalence) Suppose that we have two RCH systems $(T^*Q_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, we say them to be RCH-equivalent, or simply,*
$(T^*Q_1, \omega_1, H_1, F_1, W_1) \overset{RCH}{\sim} (T^*Q_2, \omega_2, H_2, F_2, W_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$, such that the following Hamiltonian matching conditions hold:

RHM-1: *The cotangent lift map of* φ*, that is,* $\varphi^* = T^*\varphi : T^*Q_2 \to T^*Q_1$ *is symplectic, and* $W_1 = \varphi^*(W_2)$.
RHM-2: $Im[(\mathbf{d}H_1)^\sharp + \mathrm{vlift}(F_1) - ((\varphi_*)^*\mathbf{d}H_2)^\sharp - \mathrm{vlift}(\varphi^* F_2 \varphi_*)] \subset \mathrm{vlift}(W_1)$*, where the map* $\varphi_* = (\varphi^{-1})^* : T^*Q_1 \to T^*Q_2$*, and* $(\varphi^*)_* = (\varphi_*)^* = T^*\varphi_* : T^*T^*Q_2 \to T^*T^*Q_1$*, and* Im *means the pointwise image of the map in brackets.*

It is worthy of note that our RCH system is defined by using the symplectic structure on the cotangent bundle of a configuration manifold, we must keep with the symplectic structure when we define the RCH-equivalence, that is, the induced equivalent map φ^* is symplectic on the cotangent bundle. In the same way, for the RCH systems on the symplectic fiber bundles, we can also define the RCH-equivalence by replacing T^*Q_i and $\varphi : Q_1 \to Q_2$ by E_i and $\varphi^* : E_2 \to E_1$, respectively. Moreover, the following Theorem 3.3 explains the significance of the above RCH-equivalence relation.

Theorem 3.3 *Suppose that two RCH systems* $(T^*Q_i, \omega_i, H_i, F_i, W_i)$*,* $i = 1, 2$*, are RCH-equivalent, then there exist two control laws* $u_i : T^*Q_i \to W_i$*,* $i = 1, 2$*, such that the two closed-loop systems produce the same equations of motion, that is,* $X_{(T^*Q_1,\omega_1,H_1,F_1,u_1)} \cdot \varphi^* = T(\varphi^*) X_{(T^*Q_2,\omega_2,H_2,F_2,u_2)}$*, where the map* $T(\varphi^*) : TT^*Q_2 \to TT^*Q_1$ *is the tangent map of* φ^**. Moreover, the explicit relation between the two control laws* $u_i, i = 1, 2$ *is given by*

$$\mathrm{vlift}(u_1) - \mathrm{vlift}(\varphi^* u_2 \varphi_*) = -(\mathbf{d}H_1)^\sharp - \mathrm{vlift}(F_1) + ((\varphi_*)^*\mathbf{d}H_2)^\sharp + \mathrm{vlift}(\varphi^* F_2 \varphi_*) \tag{3.2}$$

Proof: From (3.1), we have that $X_{(T^*Q_1,\omega_1,H_1,F_1,u_1)} = (\mathbf{d}H_1)^\sharp + \mathrm{vlift}(F_1) + \mathrm{vlift}(u_1)$ and

$$\begin{aligned} T(\varphi^*) X_{(T^*Q_2,\omega_2,H_2,F_2,u_2)} &= T(\varphi^*)[(\mathbf{d}H_2)^\sharp + \mathrm{vlift}(F_2) + \mathrm{vlift}(u_2)] \\ &= T(\varphi^*)(\mathbf{d}H_2)^\sharp + T(\varphi^*)\mathrm{vlift}(F_2) + T(\varphi^*)\mathrm{vlift}(u_2) \end{aligned}$$

$$\begin{array}{ccccc} T^*Q_2 & \xrightarrow{\mathrm{vlift}} & TT^*Q_2 & \xleftarrow{\sharp} & T^*T^*Q_2 \\ {\scriptstyle \varphi^*}\downarrow & & {\scriptstyle T\varphi^*}\downarrow & & {\scriptstyle (\varphi_*)^*}\downarrow \\ T^*Q_1 & \xrightarrow[\mathrm{vlift}]{} & TT^*Q_1 & \xleftarrow[\sharp]{} & T^*T^*Q_1 \end{array}$$

Diagram-2

From the commutative Diagram-2 and the definition of the vertical lift operator vlift, we have that for $\alpha \in T^*Q_2$,

$$\begin{aligned} T(\varphi^*)\mathrm{vlift}(F_2)(\alpha) &= T(\varphi^*) \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (\alpha + sF_2(\alpha)) \\ = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (\varphi^*\alpha + s\varphi^* F_2 \varphi_*(\varphi^*\alpha)) &= \mathrm{vlift}(\varphi^* F_2 \varphi_*)(\varphi^*\alpha). \end{aligned}$$

In the same way, we have that $T(\varphi^*)\mathrm{vlift}(u_2) = \mathrm{vlift}(\varphi^* u_2 \varphi_*) \cdot \varphi^*$. Since $\varphi^* : T^*Q_2 \to T^*Q_1$ is symplectic, and $i_{(\mathbf{d}H_i)^\sharp}\omega_i = \mathbf{d}H_i$, we have that $T(\varphi^*)(\mathbf{d}H_2)^\sharp = ((\varphi_*)^*\mathbf{d}H_2)^\sharp \cdot \varphi^*$. Thus,

$$T(\varphi^*) X_{(T^*Q_2,\omega_2,H_2,F_2,u_2)} = ((\varphi_*)^*\mathbf{d}H_2)^\sharp \cdot \varphi^* + \mathrm{vlift}(\varphi^* F_2 \varphi_*) \cdot \varphi^* + \mathrm{vlift}(\varphi^* u_2 \varphi_*) \cdot \varphi^*.$$

From $X_{(T^*Q_1,\omega_1,H_1,F_1,u_1)} \cdot \varphi^* = T(\varphi^*) X_{(T^*Q_2,\omega_2,H_2,F_2,u_2)}$, we have that (3.2) holds. ■

In the following we shall introduce the regular point and regular orbit reducible RCH system with symplectic form and symmetry, and show a variety of relationships of their regular reduced RCH-equivalences.

4 Regular Point Reduction of RCH Systems

Let Q be a smooth manifold and T^*Q its cotangent bundle with the symplectic form ω. Let $\Phi : G \times Q \to Q$ be a smooth left action of the Lie group G on Q, which is free and proper. Then the cotangent lifted left action $\Phi^{T^*} : G \times T^*Q \to T^*Q$ is symplectic, free and proper, and admits an Ad^*-equivariant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a regular value of $\mathbf{J}$ and $G_\mu = \{g \in G | \operatorname{Ad}_g^* \mu = \mu\}$ the isotropy subgroup of coadjoint G-action at the point μ. Since $G_\mu (\subset G)$ acts freely and properly on Q and on T^*Q, then $Q_\mu = Q/G_\mu$ is a smooth manifold and that the canonical projection $\rho_\mu : Q \to Q_\mu$ is a surjective submersion. It follows that G_μ acts also freely and properly on $\mathbf{J}^{-1}(\mu)$, so that the space $(T^*Q)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold with symplectic form ω_μ uniquely characterized by the relation

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega. \tag{4.1}$$

The map $i_\mu : \mathbf{J}^{-1}(\mu) \to T^*Q$ is the inclusion and $\pi_\mu : \mathbf{J}^{-1}(\mu) \to (T^*Q)_\mu$ is the projection. The pair $((T^*Q)_\mu, \omega_\mu)$ is called the symplectic point reduced space of (T^*Q, ω) at μ.

Remark 4.1 *If (T^*Q, ω) is a connected symplectic manifold, and $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : G \to \mathfrak{g}^*$, which is defined by $\sigma(g) := \mathbf{J}(g \cdot z) - \operatorname{Ad}_{g^{-1}}^* \mathbf{J}(z)$, where $g \in G$ and $z \in T^*Q$. Then we know that σ produces a new affine action $\Theta : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $\Theta(g, \mu) := \operatorname{Ad}_{g^{-1}}^* \mu + \sigma(g)$, where $\mu \in \mathfrak{g}^*$, with respect to which the given momentum map $\mathbf{J}$ is equivariant. Assume that G acts freely and properly on T^*Q, and $\tilde{G}_\mu$ denotes the isotropy subgroup of $\mu \in \mathfrak{g}^*$ relative to this affine action Θ and μ is a regular value of $\mathbf{J}$. Then the quotient space $(T^*Q)_\mu = \mathbf{J}^{-1}(\mu)/\tilde{G}_\mu$ is also a symplectic manifold with symplectic form ω_μ uniquely characterized by (4.1), see Ortega and Ratiu [26].*

Let $H : T^*Q \to \mathbb{R}$ be a G-invariant Hamiltonian, the flow F_t of the Hamiltonian vector field X_H leaves the connected components of $\mathbf{J}^{-1}(\mu)$ invariant and commutes with the G-action, then it induces a flow f_t^μ on $(T^*Q)_\mu$, defined by $f_t^\mu \cdot \pi_\mu = \pi_\mu \cdot F_t \cdot i_\mu$, and the vector field X_{h_μ} generated by the flow f_t^μ on $((T^*Q)_\mu, \omega_\mu)$ is Hamiltonian with the associated regular point reduced Hamiltonian function $h_\mu : (T^*Q)_\mu \to \mathbb{R}$ defined by $h_\mu \cdot \pi_\mu = H \cdot i_\mu$, and the Hamiltonian vector fields X_H and X_{h_μ} are π_μ-related. On the other hand, from section 2, we know that the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to a symplectic fiber bundle. Thus, we can introduce a regular point reducible RCH systems as follows.

Definition 4.2 *(Regular Point Reducible RCH System) A 6-tuple $(T^*Q, G, \omega, H, F, W)$, where the Hamiltonian $H : T^*Q \to \mathbb{R}$, the fiber-preserving map $F : T^*Q \to T^*Q$ and the fiber submanifold W of T^*Q are all G-invariant, is called a regular point reducible RCH system, if there exists a point $\mu \in \mathfrak{g}^*$, which is a regular value of the momentum map $\mathbf{J}$, such that the regular point reduced system, that is, the 5-tuple $((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, W_\mu)$, where $(T^*Q)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$, $\pi_\mu^* \omega_\mu = i_\mu^* \omega$, $h_\mu \cdot \pi_\mu = H \cdot i_\mu$, $f_\mu \cdot \pi_\mu = \pi_\mu \cdot F \cdot i_\mu$, $W \subset \mathbf{J}^{-1}(\mu)$, $W_\mu = \pi_\mu(W)$, is a RCH system, which is simply written as R_P-reduced RCH system. Where $((T^*Q)_\mu, \omega_\mu)$ is the R_P-reduced space, the function $h_\mu : (T^*Q)_\mu \to \mathbb{R}$ is called the reduced Hamiltonian, the fiber-preserving map $f_\mu : (T^*Q)_\mu \to (T^*Q)_\mu$ is called the reduced (external) force map, W_μ is a fiber submanifold of $(T^*Q)_\mu$ and is called the reduced control subset.*

Denote by $X_{(T^*Q,G,\omega,H,F,u)}$ the vector field of regular point reducible RCH system $(T^*Q, G, \omega, H, F, W)$ with a control law u, then

$$X_{(T^*Q,G,\omega,H,F,u)} = (\mathbf{d}H)^\sharp + \operatorname{vlift}(F) + \operatorname{vlift}(u). \tag{4.2}$$

Moreover, for the regular point reducible RCH system we can also introduce the regular point reduced controlled Hamiltonian equivalence (RpCH-equivalence) as follows.

Definition 4.3 *(RpCH-equivalence) Suppose that we have two regular point reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, we say them to be RpCH-equivalent, or simply, $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \overset{RpCH}{\sim} (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that the following Hamiltonian matching conditions hold:*
RpHM-1: *The cotangent lift map $\varphi^* : T^*Q_2 \to T^*Q_1$ is symplectic.*
RpHM-2: *For $\mu_i \in \mathfrak{g}_i^*$, the regular reducible points of RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, the map $\varphi^*_\mu = i^{-1}_{\mu_1} \cdot \varphi^* \cdot i_{\mu_2} : \mathbf{J}_2^{-1}(\mu_2) \to \mathbf{J}_1^{-1}(\mu_1)$ is $(G_{2\mu_2}, G_{1\mu_1})$-equivariant and $W_1 = \varphi^*_\mu(W_2)$, where $\mu = (\mu_1, \mu_2)$, and denote by $i^{-1}_{\mu_1}(S)$ the preimage of a subset $S \subset T^*Q_1$ for the map $i_{\mu_1} : \mathbf{J}_1^{-1}(\mu_1) \to T^*Q_1$.*
RpHM-3: $Im[(\mathbf{d}H_1)^\sharp + \mathrm{vlift}(F_1) - ((\varphi_*)^*\mathbf{d}H_2)^\sharp - \mathrm{vlift}(\varphi^* F_2 \varphi_*)] \subset \mathrm{vlift}(W_1)$.

It is worthy of note that for the regular point reducible RCH system, the induced equivalent map φ^* not only keeps the symplectic structure, but also keeps the equivariance of G-action at the regular point. If a feedback control law $u_\mu : (T^*Q)_\mu \to W_\mu$ is chosen, the R_P-reduced RCH system $((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)$ is a closed-loop regular dynamic system with a control law u_μ. Assume that its vector field $X_{((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)}$ can be expressed by

$$X_{((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)} = (\mathbf{d}h_\mu)^\sharp + \mathrm{vlift}(f_\mu) + \mathrm{vlift}(u_\mu), \tag{4.3}$$

where $(\mathbf{d}h_\mu)^\sharp = X_{h_\mu}$, $\mathrm{vlift}(f_\mu) = \mathrm{vlift}(f_\mu)X_{h_\mu}$, $\mathrm{vlift}(u_\mu) = \mathrm{vlift}(u_\mu)X_{h_\mu}$, and satisfies the condition

$$X_{((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)} \cdot \pi_\mu = T\pi_\mu \cdot X_{(T^*Q, G, \omega, H, F, u)} \cdot i_\mu. \tag{4.4}$$

Then we can obtain the following regular point reduction theorem for RCH system, which explains the relationship between the RpCH-equivalence for regular point reducible RCH systems with symmetry and the RCH-equivalence for associated R_P-reduced RCH systems. This theorem can be regarded as an extension of regular point reduction theorem of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions.

Theorem 4.4 *Two regular point reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, are RpCH-equivalent if and only if the associated R_P-reduced RCH systems $((T^*Q_i)_{\mu_i}, \omega_{i\mu_i}, h_{i\mu_i}, f_{i\mu_i}, W_{i\mu_i}), i = 1, 2$, are RCH-equivalent.*

Proof: If $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \overset{RpCH}{\sim} (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, then there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that $\varphi^* : T^*Q_2 \to T^*Q_1$ is symplectic and for $\mu_i \in \mathfrak{g}_i^*, i = 1, 2$, $\varphi^*_\mu = i^{-1}_{\mu_1} \cdot \varphi^* \cdot i_{\mu_2} : \mathbf{J}_2^{-1}(\mu_2) \to \mathbf{J}_1^{-1}(\mu_1)$ is $(G_{2\mu_2}, G_{1\mu_1})$-equivariant, $W_1 = \varphi^*_\mu(W_2)$ and RpHM-3 holds. From the following commutative Diagram-3:

$$\begin{array}{ccccc}
T^*Q_2 & \xleftarrow{i_{\mu_2}} & \mathbf{J}_2^{-1}(\mu_2) & \xrightarrow{\pi_{\mu_2}} & (T^*Q_2)_{\mu_2} \\
\varphi^* \downarrow & & \varphi^*_\mu \downarrow & & \varphi^*_{\mu/G} \downarrow \\
T^*Q_1 & \xleftarrow{i_{\mu_1}} & \mathbf{J}_1^{-1}(\mu_1) & \xrightarrow{\pi_{\mu_1}} & (T^*Q_1)_{\mu_1}
\end{array}$$

Diagram-3

We can define a map $\varphi^*_{\mu/G} : (T^*Q_2)_{\mu_2} \to (T^*Q_1)_{\mu_1}$ such that $\varphi^*_{\mu/G} \cdot \pi_{\mu_2} = \pi_{\mu_1} \cdot \varphi^*_\mu$. Because $\varphi^*_\mu : \mathbf{J}_2^{-1}(\mu_2) \to \mathbf{J}_1^{-1}(\mu_1)$ is $(G_{2\mu_2}, G_{1\mu_1})$-equivariant, $\varphi^*_{\mu/G}$ is well-defined. We shall show that

$\varphi^*_{\mu/G}$ is symplectic and $W_{1\mu_1}=\varphi^*_{\mu/G}(W_{2\mu_2})$. In fact, since $\varphi^*: T^*Q_2\to T^*Q_1$ is symplectic, the map $(\varphi^*)^*:\Omega^2(T^*Q_1)\to\Omega^2(T^*Q_2)$ satisfies $(\varphi^*)^*\omega_1=\omega_2$. By (4.1), $i^*_{\mu_i}\omega_i=\pi^*_{\mu_i}\omega_{i\mu_i}, i=1,2,$ from the following commutative Diagram-4,

$$\begin{array}{ccccc}
\Omega^2(T^*Q_1) & \xrightarrow{i^*_{\mu_1}} & \Omega^2(\mathbf{J}_1^{-1}(\mu_1)) & \xleftarrow{\pi^*_{\mu_1}} & \Omega^2((T^*Q_1)_{\mu_1})\\
{\scriptstyle (\varphi^*)^*}\downarrow & & {\scriptstyle (\varphi^*_\mu)^*}\downarrow & & {\scriptstyle (\varphi^*_{\mu/G})^*}\downarrow\\
\Omega^2(T^*Q_2) & \xrightarrow{i^*_{\mu_2}} & \Omega^2(\mathbf{J}_2^{-1}(\mu_2)) & \xleftarrow{\pi^*_{\mu_2}} & \Omega^2((T^*Q_2)_{\mu_2})
\end{array}$$

Diagram-4

we have that

$$\begin{aligned}\pi^*_{\mu_2}\cdot(\varphi^*_{\mu/G})^*\omega_{1\mu_1}&=(\varphi^*_{\mu/G}\cdot\pi_{\mu_2})^*\omega_{1\mu_1}=(\pi_{\mu_1}\cdot\varphi^*_\mu)^*\omega_{1\mu_1}=(i^{-1}_{\mu_1}\cdot\varphi^*\cdot i_{\mu_2})^*\cdot\pi^*_{\mu_1}\omega_{1\mu_1}\\
&=i^*_{\mu_2}\cdot(\varphi^*)^*\cdot(i^{-1}_{\mu_1})^*\cdot i^*_{\mu_1}\omega_1=i^*_{\mu_2}\cdot(\varphi^*)^*\omega_1=i^*_{\mu_2}\omega_2=\pi^*_{\mu_2}\omega_{2\mu_2}.\end{aligned}$$

Notice that $\pi^*_{\mu_2}$ is a surjective, thus, $(\varphi^*_{\mu/G})^*\omega_{1\mu_1}=\omega_{2\mu_2}$. Because by hypothesis $W_i\subset\mathbf{J}_i^{-1}(\mu_i)$, $W_{i\mu_i}=\pi_{\mu_i}(W_i),\ i=1,2$ and $W_1=\varphi^*_\mu(W_2)$, we have that

$$W_{1\mu_1}=\pi_{\mu_1}(W_1)=\pi_{\mu_1}\cdot\varphi^*_\mu(W_2)=\varphi^*_{\mu/G}\cdot\pi_{\mu_2}(W_2)=\varphi^*_{\mu/G}(W_{2\mu_2}).$$

Next, from (4.2) and (4.3), we know that for $i=1,2,$

$$X_{(T^*Q_i,G_i,\omega_i,H_i,F_i,u_i)}=(\mathbf{d}H_i)^\sharp+\mathrm{vlift}(F_i)+\mathrm{vlift}(u_i),$$

$$X_{((T^*Q_i)_{\mu_i},\omega_{i\mu_i},h_{i\mu_i},f_{i\mu_i},u_{i\mu_i})}=(\mathbf{d}h_{i\mu_i})^\sharp+\mathrm{vlift}(f_{i\mu_i})+\mathrm{vlift}(u_{i\mu_i}),$$

and from (4.4), we have that

$$X_{((T^*Q_i)_{\mu_i},\omega_{i\mu_i},h_{i\mu_i},f_{i\mu_i},u_{i\mu_i})}\cdot\pi_{\mu_i}=T\pi_{\mu_i}\cdot X_{(T^*Q_i,G_i,\omega_i,H_i,F_i,u_i)}\cdot i_{\mu_i}.$$

Since H_i, F_i and W_i are all G_i-invariant, $i=1,2$ and

$$h_{i\mu_i}\cdot\pi_{\mu_i}=H_i\cdot i_{\mu_i},\ \ f_{i\mu_i}\cdot\pi_{\mu_i}=\pi_{\mu_i}\cdot F_i\cdot i_{\mu_i},\ \ u_{i\mu_i}\cdot\pi_{\mu_i}=\pi_{\mu_i}\cdot u_i\cdot i_{\mu_i},\ \ i=1,2.$$

From the following commutative Diagram-5,

$$\begin{array}{ccccc}
T^*T^*Q_2 & \xrightarrow{i^*_{\mu_2}} & T^*\mathbf{J}_2^{-1}(\mu_2) & \xleftarrow{\pi^*_{\mu_2}} & T^*((T^*Q_2)_{\mu_2})\\
{\scriptstyle (\varphi^*)_*}\downarrow & & {\scriptstyle (\varphi^*_\mu)_*}\downarrow & & {\scriptstyle (\varphi^*_{\mu/G})_*}\downarrow\\
T^*T^*Q_1 & \xrightarrow{i^*_{\mu_1}} & T^*\mathbf{J}_1^{-1}(\mu_1) & \xleftarrow{\pi^*_{\mu_1}} & T^*((T^*Q_1)_{\mu_1})
\end{array}$$

Diagram-5

we have that $\pi^*_{\mu_1}\cdot(\varphi^*_{\mu/G})_*\mathbf{d}h_{2\mu_2}=i^*_{\mu_1}\cdot(\varphi^*)_*\mathbf{d}H_2$, then

$$((\varphi^*_{\mu/G})_*\mathbf{d}h_{2\mu_2})^\sharp\cdot\pi_{\mu_1}=T\pi_{\mu_1}\cdot((\varphi^*)_*\mathbf{d}H_2)^\sharp\cdot i_{\mu_1},$$

$$\mathrm{vlift}(\varphi^*_{\mu/G}\cdot f_{2\mu_2}\cdot\varphi_{\mu/G*})\cdot\pi_{\mu_1}=T\pi_{\mu_1}\cdot\mathrm{vlift}(\varphi^*F_2\varphi_*)\cdot i_{\mu_1},$$

$$\mathrm{vlift}(\varphi^*_{\mu/G}\cdot u_{2\mu_2}\cdot\varphi_{\mu/G*})\cdot\pi_{\mu_1}=T\pi_{\mu_1}\cdot\mathrm{vlift}(\varphi^*u_2\varphi_*)\cdot i_{\mu_1},$$

where $\varphi_{\mu/G*}=(\varphi^{-1})^*_{\mu/G}:(T^*Q_1)_{\mu_1}\to (T^*Q_2)_{\mu_2}$ and $(\varphi^*_{\mu/G})_*=(\varphi_{\mu/G*})^*:T^*((T^*Q_2)_{\mu_2})\to T^*((T^*Q_1)_{\mu_1})$. From Hamiltonian matching condition RpHM-3 we have that

$$Im[(\mathrm{d}h_{1\mu_1})^\sharp+\mathrm{vlift}(f_{1\mu_1})-((\varphi^*_{\mu/G})_*\mathrm{d}h_{2\mu_2})^\sharp-\mathrm{vlift}(\varphi^*_{\mu/G}\cdot f_{2\mu_2}\cdot\varphi_{\mu/G*})]\subset \mathrm{vlift}(W_{1\mu_1}). \quad (4.5)$$

So,

$$((T^*Q_1)_{\mu_1},\omega_{1\mu_1},h_{1\mu_1},f_{1\mu_1},W_{1\mu_1})\overset{RCH}{\sim}((T^*Q_2)_{\mu_2},\omega_{2\mu_2},h_{2\mu_2},f_{2\mu_2},W_{2\mu_2}).$$

Conversely, assume that R_P-reduced RCH systems $((T^*Q_i)_{\mu_i},\omega_{i\mu_i},h_{i\mu_i},f_{i\mu_i},W_{i\mu_i})$, $i=1,2$, are RCH-equivalent. Then there exists a diffeomorphism $\varphi^*_{\mu/G}:(T^*Q_2)_{\mu_2}\to (T^*Q_1)_{\mu_1}$, which is symplectic, $W_{1\mu_1}=\varphi^*_{\mu/G}(W_{2\mu_2})$, $\mu_i\in\mathfrak{g}^*_i$, $i=1,2$ and (4.5) holds. We can define a map $\varphi^*_\mu:\mathbf{J}_2^{-1}(\mu_2)\to\mathbf{J}_1^{-1}(\mu_1)$ such that $\pi_{\mu_1}\cdot\varphi^*_\mu=\varphi^*_{\mu/G}\cdot\pi_{\mu_2}$; and the map $\varphi^*:T^*Q_2\to T^*Q_1$ such that $\varphi^*\cdot i_{\mu_2}=i_{\mu_1}\cdot\varphi^*_\mu$; see the commutative Diagram-3, as well as a diffeomorphism $\varphi:Q_1\to Q_2$, whose cotangent lift is just $\varphi^*:T^*Q_2\to T^*Q_1$. From definition of φ^*_μ, we know that φ^*_μ is $(G_{2\mu_2},G_{1\mu_1})$-equivariant. In fact, for any $z_i\in\mathbf{J}_i^{-1}(\mu_i)$, $g_i\in G_{i\mu_i}$, $i=1,2$ such that $z_1=\varphi^*_\mu(z_2)$, $[z_1]=\varphi^*_{\mu/G}[z_2]$, then we have that

$$\begin{aligned}\pi_{\mu_1}\cdot\varphi^*_\mu(\Phi_{2g_2}(z_2))&=\pi_{\mu_1}\cdot\varphi^*_\mu(g_2z_2)=\varphi^*_{\mu/G}\cdot\pi_{\mu_2}(g_2z_2)=\varphi^*_{\mu/G}[z_2]=[z_1]\\&=\pi_{\mu_1}(g_1z_1)=\pi_{\mu_1}(\Phi_{1g_1}(z_1))=\pi_{\mu_1}\cdot\Phi_{1g_1}\cdot\varphi^*_\mu(z_2).\end{aligned}$$

Since π_{μ_1} is surjective, so, $\varphi^*_\mu\cdot\Phi_{2g_2}=\Phi_{1g_1}\cdot\varphi^*_\mu$. Moreover, $\pi_{\mu_1}(W_1)=W_{1\mu_1}=\varphi^*_{\mu/G}(W_{2\mu_2})=\varphi^*_{\mu/G}\cdot\pi_{2\mu_2}(W_2)=\pi_{\mu_1}\cdot\varphi^*_\mu(W_2)$. Since $W_i\subset\mathbf{J}_i^{-1}(\mu_i), i=1,2$ and π_{μ_1} is surjective, then $W_1=\varphi^*_\mu(W_2)$. We shall show that φ^* is symplectic. Because $\varphi^*_{\mu/G}:(T^*Q_2)_{\mu_2}\to(T^*Q_1)_{\mu_1}$ is symplectic, the map $(\varphi^*_{\mu/G})^*:\Omega^2((T^*Q_1)_{\mu_1})\to\Omega^2((T^*Q_2)_{\mu_2})$ satisfies $(\varphi^*_{\mu/G})^*\omega_{1\mu_1}=\omega_{2\mu_2}$. By (4.1), $i^*_{\mu_i}\omega_i=\pi^*_{\mu_i}\omega_{i\mu_i}, i=1,2$, from the commutative Diagram-4, we have that

$$\begin{aligned}i^*_{\mu_2}\omega_2&=\pi^*_{\mu_2}\omega_{2\mu_2}=\pi^*_{\mu_2}\cdot(\varphi^*_{\mu/G})^*\omega_{1\mu_1}=(\varphi^*_{\mu/G}\cdot\pi_{\mu_2})^*\omega_{1\mu_1}=(\pi_{\mu_1}\cdot\varphi^*_\mu)^*\omega_{1\mu_1}\\&=(i^{-1}_{\mu_1}\cdot\varphi^*\cdot i_{\mu_2})^*\cdot\pi^*_{\mu_1}\omega_{1\mu_1}=i^*_{\mu_2}\cdot(\varphi^*)^*\cdot(i^{-1}_{\mu_1})^*\cdot i^*_{\mu_1}\omega_1=i^*_{\mu_2}\cdot(\varphi^*)^*\omega_1.\end{aligned}$$

Notice that $i^*_{\mu_2}$ is injective, thus, $\omega_2=(\varphi^*)^*\omega_1$. Since the vector field $X_{(T^*Q_i,G_i,\omega_i,H_i,F_i,u_i)}$ and $X_{((T^*Q_i)_{\mu_i},\omega_{i\mu_i},h_{i\mu_i},f_{i\mu_i},u_{i\mu_i})}$ is π_{μ_i}-related, $i=1,2$, and H_i, F_i and W_i are all G_i-invariant, $i=1,2$, in the same way, from (4.5), we have that

$$Im[(\mathbf{d}H_1)^\sharp+\mathrm{vlift}(F_1)-((\varphi_*)^*\mathbf{d}H_2)^\sharp-\mathrm{vlift}(\varphi^*F_2\varphi_*)]\subset\mathrm{vlift}(W_1),$$

that is, Hamiltonian matching condition RpHM-3 holds. Thus,

$$(T^*Q_1,G_1,\omega_1,H_1,F_1,W_1)\overset{RpCH}{\sim}(T^*Q_2,G_2,\omega_2,H_2,F_2,W_2). \quad \blacksquare$$

Remark 4.5 *If (T^*Q,ω) is a connected symplectic manifold, and $\mathbf{J}:T^*Q\to\mathfrak{g}^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma:G\to\mathfrak{g}^*$, in this case, we can also define the regular point reducible RCH system (T^*Q,G,ω,H,F,W) and RpCH-equivalence, and prove the regular point reduction theorem for RCH system by using the above same way, where the reduced space $((T^*Q)_\mu,\omega_\mu)$ is determined by the affine action given in Remark 4.1.*

5 Regular Orbit Reduction of RCH Systems

Let $\mu \in \mathfrak{g}^*$ be a regular value of the momentum map $\mathbf{J}$ and $\mathcal{O}_\mu = G \cdot \mu \subset \mathfrak{g}^*$ be the G-orbit of the coadjoint G-action through the point μ. Since G acts freely, properly and symplectically on T^*Q, then the quotient space $(T^*Q)_{\mathcal{O}_\mu} = \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by the relation

$$i_{\mathcal{O}_\mu}^* \omega = \pi_{\mathcal{O}_\mu}^* \omega_{\mathcal{O}_\mu} + \mathbf{J}_{\mathcal{O}_\mu}^* \omega_{\mathcal{O}_\mu}^+, \tag{5.1}$$

where $\mathbf{J}_{\mathcal{O}_\mu}$ is the restriction of the momentum map $\mathbf{J}$ to $\mathbf{J}^{-1}(\mathcal{O}_\mu)$, that is, $\mathbf{J}_{\mathcal{O}_\mu} = \mathbf{J} \cdot i_{\mathcal{O}_\mu}$ and $\omega_{\mathcal{O}_\mu}^+$ is the $(+)$-symplectic structure on the orbit $\mathcal{O}_\mu$ given by

$$\omega_{\mathcal{O}_\mu}^+(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) =< \nu, [\xi, \eta] >, \ \ \forall\, \nu \in \mathcal{O}_\mu, \ \xi, \eta \in \mathfrak{g}. \tag{5.2}$$

The maps $i_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to T^*Q$ and $\pi_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to (T^*Q)_{\mathcal{O}_\mu}$ are natural injection and the projection, respectively. The pair $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is called the symplectic orbit reduced space of (T^*Q, ω).

Remark 5.1 *If (T^*Q, ω) is a connected symplectic manifold, and $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : G \to \mathfrak{g}^*$, which is defined by $\sigma(g) := \mathbf{J}(g \cdot z) - \mathrm{Ad}_{g^{-1}}^* \mathbf{J}(z)$, where $g \in G$ and $z \in T^*Q$. Then we know that σ produces a new affine action $\Theta : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $\Theta(g, \mu) := \mathrm{Ad}_{g^{-1}}^* \mu + \sigma(g)$, where $\mu \in \mathfrak{g}^*$, with respect to which the given momentum map $\mathbf{J}$ is equivariant. Assume that G acts freely and properly on T^*Q, and $\mathcal{O}_\mu = G \cdot \mu \subset \mathfrak{g}^*$ denotes the G-orbit of the point $\mu \in \mathfrak{g}^*$ with respect to this affine action Θ, and μ is a regular value of $\mathbf{J}$. Then the quotient space $(T^*Q)_{\mathcal{O}_\mu} = \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is also a symplectic manifold with symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by* (5.1)*, see Ortega and Ratiu [26].*

Let $H : T^*Q \to \mathbb{R}$ be a G-invariant Hamiltonian, the flow F_t of the Hamiltonian vector field X_H leaves the connected components of $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ invariant and commutes with the G-action, then it induces a flow $f_t^{\mathcal{O}_\mu}$ on $(T^*Q)_{\mathcal{O}_\mu}$, defined by $f_t^{\mathcal{O}_\mu} \cdot \pi_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu} \cdot F_t \cdot i_{\mathcal{O}_\mu}$, and the vector field $X_{h_{\mathcal{O}_\mu}}$ generated by the flow $f_t^{\mathcal{O}_\mu}$ on $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is Hamiltonian with the associated regular orbit reduced Hamiltonian function $h_{\mathcal{O}_\mu} : (T^*Q)_{\mathcal{O}_\mu} \to \mathbb{R}$ defined by $h_{\mathcal{O}_\mu} \cdot \pi_{\mathcal{O}_\mu} = H \cdot i_{\mathcal{O}_\mu}$ and the Hamiltonian vector fields X_H and $X_{h_{\mathcal{O}_\mu}}$ are $\pi_{\mathcal{O}_\mu}$-related.

When $Q = G$ is a Lie group with Lie algebra $\mathfrak{g}$, and the G-action is the cotangent lift of left translation, then the associated momentum map $\mathbf{J}_L : T^*G \to \mathfrak{g}^*$ is right invariant. In the same way, the momentum map $\mathbf{J}_R : T^*G \to \mathfrak{g}^*$ for the cotangent lift of right translation is left invariant. For regular value $\mu \in \mathfrak{g}^*$, $\mathcal{O}_\mu = G \cdot \mu = \{\mathrm{Ad}_{g^{-1}}^* \mu | g \in G\}$ and the Kostant-Kirilllov-Sourian (KKS) symplectic forms on coadjoint orbit $\mathcal{O}_\mu (\subset \mathfrak{g}^*)$ are given by

$$\omega_{\mathcal{O}_\mu}^-(\nu)(\mathrm{ad}_\xi^*(\nu), \mathrm{ad}_\eta^*(\nu)) = - < \nu, [\xi, \eta] >, \ \ \forall\, \nu \in \mathcal{O}_\mu, \ \xi, \eta \in \mathfrak{g}.$$

From Ortega and Ratiu [26], we know that by using the momentum map $\mathbf{J}_R$ one can induce a symplectic diffeomorphism from the symplectic point reduced space $((T^*G)_\mu, \omega_\mu)$ to the symplectic orbit space $(\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}^-)$. In general case, we maybe thought that the structure of the symplectic orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is more complex than that of the symplectic point reduced space $((T^*Q)_\mu, \omega_\mu)$, but, from the regular reduction diagram, we know that the regular orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is symplectically diffeomorphic to the regular point reduced space $((T^*Q)_\mu, \omega_\mu)$, and hence is also symplectically diffeomorphic to a symplectic fiber bundle. Thus, we can introduce a kind of the regular orbit reducible RCH systems as follows.

Definition 5.2 *(Regular Orbit Reducible RCH System) A 6-tuple* $(T^*Q, G, \omega, H, F, W)$*, where the Hamiltonian* $H : T^*Q \to \mathbb{R}$*, the fiber-preserving map* $F : T^*Q \to T^*Q$ *and the fiber submanifold* W *of* T^*Q *are all* G*-invariant, is called a regular orbit reducible RCH system, if there exists a orbit* $\mathcal{O}_\mu$*,* $\mu \in \mathfrak{g}^*$*, where* μ *is a regular value of the momentum map* $\mathbf{J}$*, such that the regular orbit reduced system, that is, the 5-tuple* $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu}, f_{\mathcal{O}_\mu}, W_{\mathcal{O}_\mu})$*, where* $(T^*Q)_{\mathcal{O}_\mu} = \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$*,* $\pi^*_{\mathcal{O}_\mu}\omega_{\mathcal{O}_\mu} = i^*_{\mathcal{O}_\mu}\omega - \mathbf{J}^*_{\mathcal{O}_\mu}\omega^+_{\mathcal{O}_\mu}$*,* $h_{\mathcal{O}_\mu} \cdot \pi_{\mathcal{O}_\mu} = H \cdot i_{\mathcal{O}_\mu}$*,* $f_{\mathcal{O}_\mu} \cdot \pi_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu} \cdot F \cdot i_{\mathcal{O}_\mu}$*,* $W \subset \mathbf{J}^{-1}(\mathcal{O}_\mu)$*,* $W_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu}(W)$*, is a RCH system, which is simply written as* R_O*-reduced RCH system. Where* $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ *is the* R_O*-reduced space, the function* $h_{\mathcal{O}_\mu} : (T^*Q)_{\mathcal{O}_\mu} \to \mathbb{R}$ *is called the reduced Hamiltonian, the fiber-preserving map* $f_{\mathcal{O}_\mu} : (T^*Q)_{\mathcal{O}_\mu} \to (T^*Q)_{\mathcal{O}_\mu}$ *is called the reduced (external) force map,* $W_{\mathcal{O}_\mu}$ *is a fiber submanifold of* $(T^*Q)_{\mathcal{O}_\mu}$*, and is called the reduced control subset.*

Denote by $X_{(T^*Q,G,\omega,H,F,u)}$ the vector field of the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$ with a control law u, then

$$X_{(T^*Q,G,\omega,H,F,u)} = (\mathbf{d}H)^\sharp + \mathrm{vlift}(F) + \mathrm{vlift}(u). \tag{5.3}$$

Moreover, for the regular orbit reducible RCH system we can also introduce the regular orbit reduced controlled Hamiltonian equivalence (RoCH-equivalence) as follows.

Definition 5.3 *(RoCH-equivalence) Suppose that we have two regular orbit reducible RCH systems* $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$*,* $i = 1, 2$*, we say them to be RoCH-equivalent, or simply,* $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \overset{RoCH}{\sim} (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$*, if there exists a diffeomorphism* $\varphi : Q_1 \to Q_2$ *such that the following Hamiltonian matching conditions hold:*
RoHM-1: *The cotangent lift map* $\varphi^* : T^*Q_2 \to T^*Q_1$ *is symplectic.*
RoHM-2: *For* $\mathcal{O}_{\mu_i}$*,* $\mu_i \in \mathfrak{g}^*_i$*, the regular reducible orbits of RCH systems* $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$*,* $i = 1, 2$*, the map* $\varphi^*_{\mathcal{O}_\mu} = i^{-1}_{\mathcal{O}_{\mu_1}} \cdot \varphi^* \cdot i_{\mathcal{O}_{\mu_2}} : \mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) \to \mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1})$ *is* (G_2, G_1)*-equivariant,* $W_1 = \varphi^*_{\mathcal{O}_\mu}(W_2)$*, and* $\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}} = (\varphi^*_{\mathcal{O}_\mu})^* \cdot \mathbf{J}^*_{1\mathcal{O}_{\mu_1}}\omega^+_{1\mathcal{O}_{\mu_1}}$*, where* $\mu = (\mu_1, \mu_2)$*, and denote by* $i^{-1}_{\mathcal{O}_{\mu_1}}(S)$ *the preimage of a subset* $S \subset T^*Q_1$ *for the map* $i_{\mathcal{O}_{\mu_1}} : \mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1}) \to T^*Q_1$*.*
RoHM-3: $Im[(\mathbf{d}H_1)^\sharp + \mathrm{vlift}(F_1) - ((\varphi_*)^*\mathbf{d}H_2)^\sharp - \mathrm{vlift}(\varphi^* F_2 \varphi_*)] \subset \mathrm{vlift}(W_1)$.

It is worthy of note that for the regular orbit reducible RCH system, the induced equivalent map φ^* not only keeps the symplectic structure and the restriction of the $(+)$-symplectic structure on the regular orbit to $\mathbf{J}^{-1}(\mathcal{O}_\mu)$, but also keeps the equivariance of G-action on the regular orbit. If a feedback control law $u_{\mathcal{O}_\mu} : (T^*Q)_{\mathcal{O}_\mu} \to W_{\mathcal{O}_\mu}$ is chosen, the R_O-reduced RCH system $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu}, f_{\mathcal{O}_\mu}, u_{\mathcal{O}_\mu})$ is a closed-loop regular dynamic system with a control law $u_{\mathcal{O}_\mu}$. Assume that its vector field $X_{((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu}, f_{\mathcal{O}_\mu}, u_{\mathcal{O}_\mu})}$ can be expressed by

$$X_{((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu}, f_{\mathcal{O}_\mu}, u_{\mathcal{O}_\mu})} = (\mathbf{d}h_{\mathcal{O}_\mu})^\sharp + \mathrm{vlift}(f_{\mathcal{O}_\mu}) + \mathrm{vlift}(u_{\mathcal{O}_\mu}), \tag{5.4}$$

where $(\mathbf{d}h_{\mathcal{O}_\mu})^\sharp = X_{h_{\mathcal{O}_\mu}}$, $\mathrm{vlift}(f_{\mathcal{O}_\mu}) = \mathrm{vlift}(f_{\mathcal{O}_\mu})X_{h_{\mathcal{O}_\mu}}$, $\mathrm{vlift}(u_{\mathcal{O}_\mu}) = \mathrm{vlift}(u_{\mathcal{O}_\mu})X_{h_{\mathcal{O}_\mu}}$, and satisfies the condition

$$X_{((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu}, f_{\mathcal{O}_\mu}, u_{\mathcal{O}_\mu})} \cdot \pi_{\mathcal{O}_\mu} = T\pi_{\mathcal{O}_\mu} \cdot X_{(T^*Q,G,\omega,H,F,u)} \cdot i_{\mathcal{O}_\mu}. \tag{5.5}$$

Then we can obtain the following regular orbit reduction theorem for RCH system, which explains the relationship between the RoCH-equivalence for regular orbit reducible RCH systems with symmetry and the RCH-equivalence for associated R_O-reduced RCH systems. This theorem can be regarded as an extension of regular orbit reduction theorem of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions.

Theorem 5.4 *If two regular orbit reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, are RoCH-equivalent, then their associated R_O-reduced RCH systems $((T^*Q)_{\mathcal{O}_{\mu_i}}, \omega_{i\mathcal{O}_{\mu_i}}, h_{i\mathcal{O}_{\mu_i}}, f_{i\mathcal{O}_{\mu_i}}, W_{i\mathcal{O}_{\mu_i}})$, $i = 1, 2$, must be RCH-equivalent. Conversely, if R_O-reduced RCH systems $((T^*Q)_{\mathcal{O}_{\mu_i}}, \omega_{i\mathcal{O}_{\mu_i}}, h_{i\mathcal{O}_{\mu_i}}, f_{i\mathcal{O}_{\mu_i}}, W_{i\mathcal{O}_{\mu_i}})$, $i = 1, 2$, are RCH-equivalent and the induced map $\varphi^*_{\mathcal{O}_\mu} : \mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) \to \mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1})$, such that $\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}} = (\varphi^*_{\mathcal{O}_\mu})^* \cdot \mathbf{J}^*_{1\mathcal{O}_{\mu_1}}\omega^+_{1\mathcal{O}_{\mu_1}}$, then the regular orbit reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, are RoCH-equivalent.*

Proof: If $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \overset{RoCH}{\sim} (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, then there exists a diffeomorphism $\varphi : Q_1 \to Q_2$, such that $\varphi^* : T^*Q_2 \to T^*Q_1$ is symplectic and for $\mu_i \in \mathfrak{g}_i^*, i = 1, 2$, $\varphi^*_{\mathcal{O}_\mu} = i^{-1}_{\mathcal{O}_{\mu_1}} \cdot \varphi^* \cdot i_{\mathcal{O}_{\mu_2}} : \mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) \to \mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1})$ is (G_2, G_1)-equivariant, $W_1 = \varphi^*_{\mathcal{O}_\mu}(W_2)$, $\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}} = (\varphi^*_{\mathcal{O}_\mu})^* \cdot \mathbf{J}^*_{1\mathcal{O}_{\mu_1}}\omega^+_{1\mathcal{O}_{\mu_1}}$, and RoHM-3 holds. From the following commutative Diagram-6,

$$\begin{CD}
T^*Q_2 @<{i_{\mathcal{O}_{\mu_2}}}<< \mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) @>{\pi_{\mathcal{O}_{\mu_2}}}>> (T^*Q_2)_{\mathcal{O}_{\mu_2}} \\
@V{\varphi^*}VV @V{\varphi^*_{\mathcal{O}_\mu}}VV @V{\varphi^*_{\mathcal{O}_\mu/G}}VV \\
T^*Q_1 @<{i_{\mathcal{O}_{\mu_1}}}<< \mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1}) @>{\pi_{\mathcal{O}_{\mu_1}}}>> (T^*Q_1)_{\mathcal{O}_{\mu_1}}
\end{CD}$$

Diagram-6

we can define a map $\varphi^*_{\mathcal{O}_\mu/G} : (T^*Q_2)_{\mathcal{O}_{\mu_2}} \to (T^*Q_1)_{\mathcal{O}_{\mu_1}}$, such that $\varphi^*_{\mathcal{O}_\mu/G} \cdot \pi_{\mathcal{O}_{\mu_2}} = \pi_{\mathcal{O}_{\mu_1}} \cdot \varphi^*_{\mathcal{O}_\mu}$. Because $\varphi^*_{\mathcal{O}_\mu} : \mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) \to \mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1})$ is (G_2, G_1)-equivariant, $\varphi^*_{\mathcal{O}_\mu/G}$ is well-defined. We can prove that $\varphi^*_{\mathcal{O}_\mu/G}$ is symplectic, that is, $(\varphi^*_{\mathcal{O}_\mu/G})^*\omega_{1\mathcal{O}_{\mu_1}} = \omega_{2\mathcal{O}_{\mu_2}}$ and $W_{1\mathcal{O}_{\mu_1}} = \varphi^*_{\mathcal{O}_\mu/G}(W_{2\mathcal{O}_{\mu_2}})$. In fact, since $\varphi^* : T^*Q_1 \to T^*Q_2$ is symplectic, the map $(\varphi^*)^* : \Omega^2(T^*Q_1) \to \Omega^2(T^*Q_2)$ satisfies $(\varphi^*)^*\omega_1 = \omega_2$. By (5.1), $i^*_{\mathcal{O}_{\mu_i}}\omega_i = \pi^*_{\mathcal{O}_{\mu_i}}\omega_{i\mathcal{O}_{\mu_i}} + \mathbf{J}^*_{i\mathcal{O}_{\mu_i}}\omega^+_{i\mathcal{O}_{\mu_i}}$, $i = 1, 2$, and $\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}} = (\varphi^*_{\mathcal{O}_\mu})^* \cdot \mathbf{J}^*_{1\mathcal{O}_{\mu_1}}\omega^+_{1\mathcal{O}_{\mu_1}}$, from the following commutative Diagram-7,

$$\begin{CD}
\Omega^2(T^*Q_1) @>{i^*_{\mathcal{O}_{\mu_1}}}>> \Omega^2(\mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1})) @<{\pi^*_{\mathcal{O}_{\mu_1}}}<< \Omega^2((T^*Q_1)_{\mathcal{O}_{\mu_1}}) \\
@V{(\varphi^*)^*}VV @V{(\varphi^*_{\mathcal{O}_\mu})^*}VV @V{(\varphi^*_{\mathcal{O}_\mu/G})^*}VV \\
\Omega^2(T^*Q_2) @>{i^*_{\mathcal{O}_{\mu_2}}}>> \Omega^2(\mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2})) @<{\pi^*_{\mathcal{O}_{\mu_2}}}<< \Omega^2((T^*Q_2)_{\mathcal{O}_{\mu_2}})
\end{CD}$$

Diagram-7

we have that

$$\begin{aligned}
&\pi^*_{\mathcal{O}_{\mu_2}} \cdot (\varphi^*_{\mathcal{O}_\mu/G})^*\omega_{1\mathcal{O}_{\mu_1}} = (\varphi^*_{\mathcal{O}_\mu/G} \cdot \pi_{\mathcal{O}_{\mu_2}})^*\omega_{1\mathcal{O}_{\mu_1}} = (\pi_{\mathcal{O}_{\mu_1}} \cdot \varphi^*_{\mathcal{O}_\mu})^*\omega_{1\mathcal{O}_{\mu_1}} \\
&= (\varphi^*_{\mathcal{O}_\mu})^* \cdot \pi^*_{\mathcal{O}_{\mu_1}}\omega_{1\mathcal{O}_{\mu_1}} = (i^{-1}_{\mathcal{O}_{\mu_1}} \cdot \varphi^* \cdot i_{\mathcal{O}_{\mu_2}})^* \cdot i^*_{\mathcal{O}_{\mu_1}}\omega_1 - (\varphi^*_{\mathcal{O}_\mu})^* \cdot \mathbf{J}^*_{1\mathcal{O}_{\mu_1}}\omega^+_{1\mathcal{O}_{\mu_1}} \\
&= i^*_{\mathcal{O}_{\mu_2}} \cdot (\varphi^*)^*\omega_1 - \mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}} = i^*_{\mathcal{O}_{\mu_2}}\omega_2 - \mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}} = \pi^*_{\mathcal{O}_{\mu_2}}\omega_{2\mathcal{O}_{\mu_2}}.
\end{aligned}$$

Because $\pi^*_{\mathcal{O}_{\mu_2}}$ is surjective, thus $(\varphi^*_{\mathcal{O}_\mu/G})^*\omega_{1\mathcal{O}_{\mu_1}} = \omega_{2\mathcal{O}_{\mu_2}}$. Notice that $W_i \subset \mathbf{J}_i^{-1}(\mathcal{O}_{\mu_i})$, $W_{i\mathcal{O}_{\mu_i}} = \pi_{\mathcal{O}_{\mu_i}}(W_i)$, $i = 1, 2$, and $W_1 = \varphi^*_{\mathcal{O}_\mu}(W_2)$, we have that

$$W_{1\mathcal{O}_{\mu_1}} = \pi_{\mathcal{O}_{\mu_1}}(W_1) = \pi_{\mathcal{O}_{\mu_1}} \cdot \varphi^*_{\mathcal{O}_\mu}(W_2) = \varphi^*_{\mathcal{O}_\mu/G} \cdot \pi_{\mathcal{O}_{\mu_2}}(W_2) = \varphi^*_{\mathcal{O}_\mu/G}(W_{2\mathcal{O}_{\mu_2}}).$$

Next, from (5.3) and (5.4), we know that for $i = 1, 2$,

$$X_{(T^*Q_i, G_i, \omega_i, H_i, F_i, u_i)} = (\mathbf{d}H_i)^\sharp + \text{vlift}(F_i) + \text{vlift}(u_i),$$

$$X_{((T^*Q_i)_{\mathcal{O}_{\mu_i}},\omega_{i\mathcal{O}_{\mu_i}},h_{i\mathcal{O}_{\mu_i}},f_{i\mathcal{O}_{\mu_i}},u_{i\mathcal{O}_{\mu_i}})} = (\mathbf{d}h_{i\mathcal{O}_{\mu_i}})^\sharp + \mathrm{vlift}(f_{i\mathcal{O}_{\mu_i}}) + \mathrm{vlift}(u_{i\mathcal{O}_{\mu_i}}),$$

and from (5.5), we have that

$$X_{((T^*Q_i)_{\mathcal{O}_{\mu_i}},\omega_{i\mathcal{O}_{\mu_i}},h_{i\mathcal{O}_{\mu_i}},f_{i\mathcal{O}_{\mu_i}},u_{i\mathcal{O}_{\mu_i}})} \cdot \pi_{\mathcal{O}_{\mu_i}} = T\pi_{\mathcal{O}_{\mu_i}} \cdot X_{(T^*Q_i,G_i,\omega_i,H_i,F_i,u_i)} \cdot i_{\mathcal{O}_{\mu_i}}.$$

Since H_i, F_i and W_i are all G_i-invariant, $i=1,2$, and

$$h_{i\mathcal{O}_{\mu_i}} \cdot \pi_{\mathcal{O}_{\mu_i}} = H_i \cdot i_{\mathcal{O}_{\mu_i}}, \;\; f_{i\mathcal{O}_{\mu_i}} \cdot \pi_{\mathcal{O}_{\mu_i}} = \pi_{\mathcal{O}_{\mu_i}} \cdot F_i \cdot i_{\mathcal{O}_{\mu_i}}, \;\; u_{i\mathcal{O}_{\mu_i}} \cdot \pi_{\mathcal{O}_{\mu_i}} = \pi_{\mathcal{O}_{\mu_i}} \cdot u_i \cdot i_{\mathcal{O}_{\mu_i}}, \;\; i=1,2.$$

From the following commutative Diagram-8,

$$\begin{array}{ccccc}
T^*T^*Q_2 & \xrightarrow{i^*_{\mathcal{O}_{\mu_2}}} & T^*\mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) & \xleftarrow{\pi^*_{\mathcal{O}_{\mu_2}}} & T^*((T^*Q_2)_{\mathcal{O}_{\mu_2}}) \\
{\scriptstyle (\varphi^*)_*}\downarrow & & {\scriptstyle (\varphi^*_{\mathcal{O}_\mu})_*}\downarrow & & {\scriptstyle (\varphi^*_{\mathcal{O}_\mu/G})_*}\downarrow \\
T^*T^*Q_1 & \xrightarrow{i^*_{\mathcal{O}_{\mu_1}}} & T^*\mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1}) & \xleftarrow{\pi^*_{\mathcal{O}_{\mu_1}}} & T^*((T^*Q_1)_{\mathcal{O}_{\mu_1}})
\end{array}$$

Diagram-8

we have that $\pi^*_{\mathcal{O}_{\mu_1}} \cdot (\varphi^*_{\mathcal{O}_\mu/G})_* \mathbf{d}h_{2\mathcal{O}_{\mu_2}} = i^*_{\mathcal{O}_{\mu_1}} \cdot (\varphi^*)_* \mathbf{d}H_2$, then

$$((\varphi^*_{\mathcal{O}_\mu/G})_* \mathbf{d}h_{2\mathcal{O}_{\mu_2}})^\sharp \cdot \pi_{\mathcal{O}_{\mu_1}} = T\pi_{\mathcal{O}_{\mu_1}} \cdot ((\varphi^*)_* \mathbf{d}H_2)^\sharp \cdot i_{\mathcal{O}_{\mu_1}},$$

$$\mathrm{vlift}(\varphi^*_{\mathcal{O}_\mu/G} \cdot f_{2\mathcal{O}_{\mu_2}} \cdot \varphi_{\mathcal{O}_\mu/G*}) \cdot \pi_{\mathcal{O}_{\mu_1}} = T\pi_{\mathcal{O}_{\mu_1}} \cdot \mathrm{vlift}(\varphi^* F_2 \varphi_*) \cdot i_{\mathcal{O}_{\mu_1}},$$

$$\mathrm{vlift}(\varphi^*_{\mathcal{O}_\mu/G} \cdot u_{2\mathcal{O}_{\mu_2}} \cdot \varphi_{\mathcal{O}_\mu/G*}) \cdot \pi_{\mathcal{O}_{\mu_1}} = T\pi_{\mathcal{O}_{\mu_1}} \cdot \mathrm{vlift}(\varphi^* u_2 \varphi_*) \cdot i_{\mathcal{O}_{\mu_1}},$$

where the map $\varphi_{\mathcal{O}_\mu/G*} = (\varphi^{-1})^*_{\mathcal{O}_\mu/G} : (T^*Q_1)_{\mathcal{O}_{\mu_1}} \to (T^*Q_2)_{\mathcal{O}_{\mu_2}}$ and $(\varphi^*_{\mathcal{O}_\mu/G})_* = (\varphi_{\mathcal{O}_\mu/G*})^* : T^*((T^*Q_2)_{\mathcal{O}_{\mu_2}}) \to T^*((T^*Q_1)_{\mathcal{O}_{\mu_1}})$. From the Hamiltonian matching condition RoHM-3 we have that

$$Im[(\mathbf{d}h_{1\mathcal{O}_{\mu_1}})^\sharp + \mathrm{vlift}(f_{1\mathcal{O}_{\mu_1}}) - ((\varphi^*_{\mathcal{O}_\mu/G})_* \mathbf{d}h_{2\mathcal{O}_{\mu_2}})^\sharp - \mathrm{vlift}(\varphi^*_{\mathcal{O}_\mu/G} \cdot f_{2\mathcal{O}_{\mu_2}} \cdot \varphi_{\mathcal{O}_\mu/G*})] \subset \mathrm{vlift}(W_{1\mathcal{O}_{\mu_1}}). \tag{5.6}$$

So,

$$((T^*Q_1)_{\mathcal{O}_{\mu_1}}, \omega_{1\mathcal{O}_{\mu_1}}, h_{1\mathcal{O}_{\mu_1}}, f_{1\mathcal{O}_{\mu_1}}, W_{1\mathcal{O}_{\mu_1}}) \overset{RCH}{\sim} ((T^*Q_2)_{\mathcal{O}_{\mu_2}}, \omega_{2\mathcal{O}_{\mu_2}}, h_{2\mathcal{O}_{\mu_2}}, f_{2\mathcal{O}_{\mu_2}}, W_{2\mathcal{O}_{\mu_2}}).$$

Conversely, assume that R_O-reduced RCH systems $((T^*Q_i)_{\mathcal{O}_{\mu_i}}, \omega_{i\mathcal{O}_{\mu_i}}, h_{i\mathcal{O}_{\mu_i}}, f_{i\mathcal{O}_{\mu_i}}, W_{i\mathcal{O}_{\mu_i}})$, $i=1,2$, are RCH-equivalent, then there exists a diffeomorphism $\varphi^*_{\mathcal{O}_\mu/G} : (T^*Q_2)_{\mathcal{O}_{\mu_2}} \to (T^*Q_1)_{\mathcal{O}_{\mu_1}}$, which is symplectic, $W_{1\mathcal{O}_{\mu_1}} = \varphi^*_{\mathcal{O}_\mu/G}(W_{2\mathcal{O}_\mu})$ and (5.6) hold. Thus, we can define a map $\varphi^*_{\mathcal{O}_\mu} : \mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) \to \mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1})$ such that $\pi_{\mathcal{O}_{\mu_1}} \cdot \varphi^*_{\mathcal{O}_\mu} = \varphi^*_{\mathcal{O}_\mu/G} \cdot \pi_{\mathcal{O}_{\mu_2}}$; and map $\varphi^* : T^*Q_2 \to T^*Q_1$ such that $i_{\mathcal{O}_{\mu_1}} \cdot \varphi^*_{\mathcal{O}_\mu} = \varphi^* \cdot i_{\mathcal{O}_{\mu_2}}$; see the commutative Diagram-6, as well as a diffeomorphism $\varphi : Q_1 \to Q_2$, whose cotangent lift is just $\varphi^* : T^*Q_2 \to T^*Q_1$. At first, from definition of $\varphi^*_{\mathcal{O}_\mu}$ we know that $\varphi^*_{\mathcal{O}_\mu}$ is (G_2, G_1)-equivariant. In fact, for any $z_i \in \mathbf{J}_i^{-1}(\mathcal{O}_{\mu_i})$, $g_i \in G_i$, $i=1,2$ such that $z_1 = \varphi^*_{\mathcal{O}_\mu}(z_2)$, $[z_1] = \varphi^*_{\mathcal{O}_\mu/G}[z_2]$, then we have that

$$\begin{aligned}
&\pi_{\mathcal{O}_{\mu_1}} \cdot \varphi^*_{\mathcal{O}_\mu}(\Phi_{2g_2}(z_2)) = \pi_{\mathcal{O}_{\mu_1}} \cdot \varphi^*_{\mathcal{O}_\mu}(g_2 z_2) = \varphi^*_{\mathcal{O}_\mu/G} \cdot \pi_{\mathcal{O}_{\mu_2}}(g_2 z_2) = \varphi^*_{\mathcal{O}_\mu/G}[z_2] \\
&= [z_1] = \pi_{\mathcal{O}_{\mu_1}}(g_1 z_1) = \pi_{\mathcal{O}_{\mu_1}}(\Phi_{1g_1}(z_1)) = \pi_{\mathcal{O}_{\mu_1}} \cdot \Phi_{1g_1} \cdot \varphi^*_{\mathcal{O}_\mu}(z_2).
\end{aligned}$$

Since $\pi_{\mathcal{O}_{\mu_1}}$ is surjective, so, $\varphi^*_{\mathcal{O}_\mu}\cdot\Phi_{2g_2}=\Phi_{1g_1}\cdot\varphi^*_{\mathcal{O}_\mu}$. Moreover, $\pi_{\mathcal{O}_{\mu_1}}(W_1)=W_{1\mathcal{O}_{\mu_1}}=\varphi^*_{\mathcal{O}_\mu/G}(W_{2\mathcal{O}_{\mu_2}})=\varphi^*_{\mathcal{O}_\mu/G}\cdot\pi_{\mathcal{O}_{\mu_2}}(W_2)=\pi_{\mathcal{O}_{\mu_1}}\cdot\varphi^*_{\mathcal{O}_\mu}(W_2)$. Since $W_i\subset\mathbf{J}_i^{-1}(\mathcal{O}_{\mu_i})$, $i=1,2$, and $\pi_{\mathcal{O}_{\mu_1}}$ is surjective, then $W_1=\varphi^*_{\mathcal{O}_\mu}(W_2)$. Now we shall show that φ^* is symplectic, that is, $\omega_2=(\varphi^*)^*\omega_1$. In fact, since $\varphi^*_{\mathcal{O}_\mu/G}:(T^*Q_2)_{\mathcal{O}_{\mu_2}}\to(T^*Q_1)_{\mathcal{O}_{\mu_1}}$ is symplectic, the map $(\varphi^*_{\mathcal{O}_\mu/G})^*:\Omega^2((T^*Q_1)_{\mathcal{O}_{\mu_1}})\to\Omega^2((T^*Q_2)_{\mathcal{O}_{\mu_2}})$ satisfies $(\varphi^*_{\mathcal{O}_\mu/G})^*\omega_{1\mathcal{O}_{\mu_1}}=\omega_{2\mathcal{O}_{\mu_2}}$. By (5.1), $i^*_{\mathcal{O}_{\mu_i}}\omega_i=\pi^*_{\mathcal{O}_{\mu_i}}\omega_{i\mathcal{O}_{\mu_i}}+\mathbf{J}^*_{i\mathcal{O}_{\mu_i}}\omega^+_{i\mathcal{O}_{\mu_i}}$, $i=1,2$, from the commutative Diagram-7, we have that

$$\begin{aligned}
&i^*_{\mathcal{O}_{\mu_2}}\omega_2=\pi^*_{\mathcal{O}_{\mu_2}}\omega_{2\mathcal{O}_{\mu_2}}+\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}}=\pi^*_{2\mathcal{O}_{\mu_2}}\cdot(\varphi^*_{\mathcal{O}_\mu/G})^*\omega_{1\mathcal{O}_{\mu_1}}+\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}}\\
&=(\varphi^*_{\mathcal{O}_\mu/G}\cdot\pi_{\mathcal{O}_{\mu_2}})^*\omega_{1\mathcal{O}_{\mu_1}}+\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}}=(\pi_{\mathcal{O}_{\mu_1}}\cdot\varphi^*_{\mathcal{O}_\mu})^*\omega_{1\mathcal{O}_{\mu_1}}+\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}}\\
&=(i^{-1}_{\mathcal{O}_{\mu_1}}\cdot\varphi^*\cdot i_{\mathcal{O}_{\mu_2}})^*\cdot\pi^*_{\mathcal{O}_{\mu_1}}\omega_{1\mathcal{O}_{\mu_1}}+\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}}\\
&=i^*_{\mathcal{O}_{\mu_2}}\cdot(\varphi^*)^*\cdot(i^{-1}_{\mathcal{O}_{\mu_1}})^*\cdot[i^*_{\mathcal{O}_{\mu_1}}\omega_1-\mathbf{J}^*_{1\mathcal{O}_{\mu_1}}\omega^+_{1\mathcal{O}_{\mu_1}}]+\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}}\\
&=i^*_{\mathcal{O}_{\mu_2}}\cdot(\varphi^*)^*\omega_1-(\varphi^*_{\mathcal{O}_\mu})^*\cdot\mathbf{J}^*_{1\mathcal{O}_{\mu_1}}\omega^+_{1\mathcal{O}_{\mu_1}}+\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}}
\end{aligned}$$

Notice that $i^*_{\mathcal{O}_{\mu_2}}$ is injective, and by our hypothesis, $\mathbf{J}^*_{2\mathcal{O}_{\mu_2}}\omega^+_{2\mathcal{O}_{\mu_2}}=(\varphi^*_{\mathcal{O}_\mu})^*\cdot\mathbf{J}^*_{1\mathcal{O}_{\mu_1}}\omega^+_{1\mathcal{O}_{\mu_1}}$, then $\omega_2=(\varphi^*)^*\omega_1$, that is, φ^* is symplectic. Since the vector fields $X_{(T^*Q_i,G_i,\omega_i,H_i,F_i,u_i)}$ and $X_{((T^*Q_i)_{\mathcal{O}_{\mu_i}},\omega_{i\mathcal{O}_{\mu_i}},h_{i\mathcal{O}_{\mu_i}},f_{i\mathcal{O}_{\mu_i}},u_{i\mathcal{O}_{\mu_i}})}$ is $\pi_{\mathcal{O}_{\mu_i}}$-related, $i=1,2$, and H_i,F_i and W_i are all G_i-invariant, $i=1,2$, in the same way, from (5.6) we have that Hamiltonian matching condition RoHM-3 holds. Thus,

$$(T^*Q_1,G_1,\omega_1,H_1,F_1,W_1)\overset{RoCH}{\sim}(T^*Q_2,G_2,\omega_2,H_2,F_2,W_2).\qquad\blacksquare$$

Remark 5.5 *If (T^*Q,ω) is a connected symplectic manifold, and $\mathbf{J}:T^*Q\to\mathfrak{g}^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma:G\to\mathfrak{g}^*$, in this case, we can also define the regular orbit reducible RCH system (T^*Q,G,ω,H,F,W) and RoCH-equivalence, and prove the regular orbit reduction theorem for RCH system by using the above same way, where the reduced space $((T^*Q)_{\mathcal{O}_\mu},\omega_{\mathcal{O}_\mu})$ is determined by the affine action given in Remark 5.1.*

6 Applications

As the applications of regular point reduction theory of RCH systems with symmetry, in this section, we first study the regular point reducible RCH systems on a Lie group and on its generalization, respectively, and give their R_P-reduced RCH systems, which are the RCH systems on a coadjoint orbit and on its generalization, respectively. Next, we describe uniformly the rigid body and heavy top, as well as them with internal rotors (or the external force torques) as the regular point reducible RCH systems on the rotation group SO(3) and on the Euclidean group SE(3), as well as on their generalizations, respectively, and give their R_P-reduced RCH systems and discuss their RCH-equivalence. Moreover, in order to understand well the abstract definition of RCH system and the significance of Theorem 3.3, we describe the RCH system from the viewpoint of port Hamiltonian system with a symplectic structure, and state the relationship between RCH-equivalence and equivalence of port Hamiltonian system.

6.1 Regular Point Reducible RCH Systems on a Lie Group and Its Generalization

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and T^*G its cotangent bundle with the canonical symplectic form ω_0. A RCH system on G is a 5-tuple (T^*G,ω_0,H,F,W), where (T^*G,ω_0,H) is a Hamiltonian system and $H : T^*G \to \mathbb{R}$ is a Hamiltonian, the fiber-preserving map $F : T^*G \to T^*G$ is a (external) force map and the fiber submanifold W of T^*G is a control subset. In the following we shall give its R_P-reduced RCH system. We know that the left and right translation on G induce the left and right action of G on itself. If $I_g : G \to G$; $I_g(h) = ghg^{-1} = L_g \cdot R_{g^{-1}}(h)$, for $g, h \in G$, is the inner automorphism on G, then the adjoint representation of a Lie group G is defined by $\mathrm{Ad}_g = T_eI_g = T_{g^{-1}}L_g \cdot T_eR_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$, and the coadjoint representation is given by $\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$; $\langle \mathrm{Ad}^*_{g^{-1}}(\mu), \xi \rangle = \langle \mu, \mathrm{Ad}_{g^{-1}}(\xi) \rangle$, where $\mu \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ and $\langle , \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Since the coadjoint representation $\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$ can induce a left coadjoint action of G on $\mathfrak{g}^*$, then the coadjoint orbit $\mathcal{O}_\mu$ of this action through $\mu \in \mathfrak{g}^*$ is an immersed submanifold of $\mathfrak{g}^*$. We know that $\mathfrak{g}^*$ is a Poisson manifold with respect to the $(\pm)$-Lie-Poisson bracket $\{\cdot, \cdot\}_\pm$ defined by

$$\{f, g\}_\pm(\mu) := \pm < \mu, [\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}] >, \;\; \forall f, g \in C^\infty(\mathfrak{g}^*), \;\; \mu \in \mathfrak{g}^*, \tag{6.1}$$

where the element $\frac{\delta f}{\delta \mu} \in \mathfrak{g}$ is defined by the equality $< v, \frac{\delta f}{\delta \mu} >:= Df(\mu) \cdot v$, for any $v \in \mathfrak{g}^*$, see Marsden and Ratiu [22]. Thus, for the coadjoint orbit $\mathcal{O}_\mu$, $\mu \in \mathfrak{g}^*$, the orbit symplectic structure can be defined by

$$\omega^\pm_{\mathcal{O}_\mu}(\nu)(\mathrm{ad}^*_\xi(\nu), \mathrm{ad}^*_\eta(\nu)) = \pm \langle \nu, [\xi, \eta] \rangle, \qquad \forall\, \xi, \eta \in \mathfrak{g}, \;\; \nu \in \mathcal{O}_\mu \subset \mathfrak{g}^*, \tag{6.2}$$

which are coincide with the restriction of the Lie-Poisson brackets on $\mathfrak{g}^*$ to the coadjoint orbit $\mathcal{O}_\mu$. From the Symplectic Stratification theorem we know that a finite dimensional Poisson manifold is the disjoint union of its symplectic leaves, and its each symplectic leaf is an injective immersed Poisson submanifold whose induced Poisson structure is symplectic. In consequence, when $\mathfrak{g}^*$ is endowed one of the Lie Poisson structures $\{\cdot, \cdot\}_\pm$, the symplectic leaves of the Poisson manifolds $(\mathfrak{g}^*, \{\cdot, \cdot\}_\pm)$ coincide with the connected components of the orbits of the elements in $\mathfrak{g}^*$ under the coadjoint action. From Abraham and Marsden [1], we have the following result.

Proposition 6.1 *The coadjoint orbit $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$, $\mu \in \mathfrak{g}^*$, is symplectically diffeomorphic to a regular point reduced space $((T^*G)_\mu, \omega_\mu)$ of T^*G.*

We now identify T^*G and $G \times \mathfrak{g}^*$ by using the left translation. In fact, the map $\lambda : T^*G \to G \times \mathfrak{g}^*$, $\lambda(\alpha_g) := (g, (T_eL_g)^*\alpha_g)$, for any $\alpha_g \in T^*_gG$, which defines a vector bundle isomorphism usually referred to as the left trivialization of T^*G. In the same way, we can also identify tangent bundle TG and $G \times \mathfrak{g}$ by using the left translation. In consequence, we can consider the Lagrangian $L(g, \xi) : TG \cong G \times \mathfrak{g} \to \mathbb{R}$, which is usual the kinetic minus the potential energy of the system, where $(g, \xi) \in G \times \mathfrak{g}$, and $\xi \in \mathfrak{g}$, regarded as the velocity of system. If we introduce the conjugate momentum $p_i = \frac{\partial L}{\partial \xi^i}$, $i = 1, \cdots, n$, $n = dimG$, and by the Legendre transformation $FL : TG \cong G \times \mathfrak{g} \to T^*G \cong G \times \mathfrak{g}^*$, $(g^i, \xi^i) \to (g^i, p_i)$, we have the Hamiltonian $H(g, p) : T^*G \cong G \times \mathfrak{g}^* \to \mathbb{R}$ given by

$$H(g^i, p_i) = \sum_{i=1}^n p_i \xi^i - L(g^i, \xi^i). \tag{6.3}$$

If the Hamiltonian $H(g, p) : T^*G \cong G \times \mathfrak{g} \to \mathbb{R}$ is left cotangent lifted G-action invariant, for $\mu \in \mathfrak{g}^*$ we have the associated reduced Hamiltonian $h_\mu : (T^*G)_\mu \cong \mathcal{O}_\mu \to \mathbb{R}$, defined by $h_\mu \cdot \pi_\mu = H \cdot i_\mu$. By the $(\pm)$-Lie-Poisson brackets on $\mathfrak{g}^*$ and the symplecitic structure on the coadjoint orbit $\mathcal{O}_\mu$, we have the associated Hamiltonian vector field X_{h_μ} given by

$$X_{h_\mu}(\nu) = \mp\, \mathrm{ad}^*_{\delta h_\mu / \delta \nu}\, \nu, \quad \forall \nu \in \mathcal{O}_\mu. \tag{6.4}$$

See Marsden and Ratiu [22]. Thus, if the Hamiltonian $H : T^*G \to \mathbb{R}$, the fiber-preserving map $F : T^*G \to T^*G$ and the fiber submanifold W of T^*G are all left cotangent lifted G-action invariant, then we may give the R_P-reduced RCH system as follows.

Theorem 6.2 *The 6-tuple $(T^*G, G, \omega_0, H, F, W)$ is a regular point reducible RCH system on Lie group G, where the Hamiltonian $H : T^*G \to \mathbb{R}$, the fiber-preserving map $F : T^*G \to T^*G$ and the fiber submanifold W of T^*G are all left cotangent lifted G-action invariant. For a point $\mu \in \mathfrak{g}^*$, the regular value of the momentum map $\mathbf{J}_L : T^*G \to \mathfrak{g}^*$, the R_P-reduced system, that is, the 5-tuple $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu}, h_\mu, f_\mu, W_\mu)$, is a RCH system, where $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is the coadjoint orbit, $\omega^-_{\mathcal{O}_\mu}$ is orbit symplectic form, $h_\mu \cdot \pi_\mu = H \cdot i_\mu$, $f_\mu \cdot \pi_\mu = \pi_\mu \cdot F \cdot i_\mu$, $W \subset \mathbf{J}_L^{-1}(\mu)$, and $W_\mu = \pi_\mu(W) \subset \mathcal{O}_\mu$. Moreover, two regular point reducible RCH system $(T^*G_i, G_i, \omega_{i0}, H_i, F_i, W_i)$, $\quad i = 1, 2$, are RpCH-equivalent if and only if the associated R_P-reduced RCH systems $(\mathcal{O}_{i\mu_i}, \omega^-_{\mathcal{O}_{i\mu_i}}, h_{i\mu_i}, f_{i\mu_i}, W_{i\mu_i})$, $i = 1, 2$, are RCH-equivalent.*

Next, in order to study the regular reduction of rigid body and heavy top with internal rotors, we need the regular symplectic reduction theory of the cotangent bundle T^*Q, where the configuration space $Q = G \times V$, and G is a Lie group and V is a k-dimensional vector space. Defined the left G-action $\Phi : G \times Q \to Q$, $\Phi(g, (h, \theta)) := (gh, \theta)$, for any $g, h \in G$, $\theta \in V$, that is , the G-action on Q is the left translation on the first factor G, and G acts trivially on the second factor V. Because $T^*Q = T^*G \times T^*V$, and $T^*V = V \times V^*$, by using the left trivialization of T^*G, we have that $T^*Q = G \times \mathfrak{g}^* \times V \times V^*$. If the left G-action $\Phi : G \times Q \to Q$ is free and proper, then the cotangent lift of the action to its cotangent bundle T^*Q, given by $\Phi^{T^*} : G \times T^*Q \to T^*Q$, $\Phi^{T^*}(g, (h, \mu, \theta, \lambda)) := (gh, \mu, \theta, \lambda)$, for any $g, h \in G$, $\mu \in \mathfrak{g}^*$, $\theta \in V$, $\lambda \in V^*$, is also a free and proper action, and the orbit space $(T^*Q)/G$ is a smooth manifold and $\pi : T^*Q \to (T^*Q)/G$ is a smooth submersion. Since G acts trivially on $\mathfrak{g}^*$, V and V^*, it follows that $(T^*Q)/G$ is diffeomorphic to $\mathfrak{g}^* \times V \times V^*$.

For $\mu \in \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$ has the orbit symplectic forms $\omega^\pm_{\mathcal{O}_\mu}$. Let ω_V be the canonical symplectic form on $T^*V \cong V \times V^*$ given by

$$\omega_V((\theta_1, \lambda_1), (\theta_2, \lambda_2)) =< \lambda_2, \theta_1 > - < \lambda_1, \theta_2 >,$$

where $(\theta_i, \lambda_i) \in V \times V^*$, $i = 1, 2$, $< \cdot, \cdot >$ is the natural pairing between V^* and V. Thus, we can induce a symplectic forms $\tilde{\omega}^\pm_{\mathcal{O}_\mu \times V \times V^*} = \pi^*_{\mathcal{O}_\mu} \omega^\pm_{\mathcal{O}_\mu} + \pi^*_V \omega_V$ on the smooth manifold $\mathcal{O}_\mu \times V \times V^*$, where the maps $\pi_{\mathcal{O}_\mu} : \mathcal{O}_\mu \times V \times V^* \to \mathcal{O}_\mu$ and $\pi_V : \mathcal{O}_\mu \times V \times V^* \to V \times V^*$ are canonical projections. On the other hand, from $T^*Q = T^*G \times T^*V$ we know that there is a canonical symplectic form $\omega_Q = \pi_1^* \omega_0 + \pi_2^* \omega_V$ on T^*Q, where ω_0 is the canonical symplectic form on T^*G and the maps $\pi_1 : Q = G \times V \to G$ and $\pi_2 : Q = G \times V \to V$ are canonical projections. Then the cotangent lift of the left G-action $\Phi^{T^*} : G \times T^*Q \to T^*Q$ is also symplectic, and admits an associated Ad^*-equivariant momentum map $\mathbf{J}_Q : T^*Q \to \mathfrak{g}^*$ such that $\mathbf{J}_Q \cdot \pi_1^* = \mathbf{J}_L$, where $\mathbf{J}_L : T^*G \to \mathfrak{g}^*$ is a momentum map of left G-action on T^*G, and $\pi_1^* : T^*G \to T^*Q$. If $\mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}_Q$, then $\mu \in \mathfrak{g}^*$ is also a regular value of $\mathbf{J}_L$ and $\mathbf{J}_Q^{-1}(\mu) \cong \mathbf{J}_L^{-1}(\mu) \times V \times V^*$. Denote by $G_\mu = \{g \in G | \, \mathrm{Ad}^*_g \mu = \mu\}$ the isotropy subgroup of coadjoint G-action at the point $\mu \in \mathfrak{g}^*$. It follows that G_μ acts also freely and properly on $\mathbf{J}_Q^{-1}(\mu)$, the regular point reduced space $(T^*Q)_\mu = \mathbf{J}_Q^{-1}(\mu)/G_\mu \cong (T^*G)_\mu \times V \times V^*$ of (T^*Q, ω_Q) at μ, is a symplectic manifold with symplectic form ω_μ uniquely characterized by the relation $\pi_\mu^* \omega_\mu = i_\mu^* \omega_Q = i_\mu^* \pi_1^* \omega_0 + i_\mu^* \pi_2^* \omega_V$, where the map $i_\mu : \mathbf{J}_Q^{-1}(\mu) \to T^*Q$ is the inclusion and $\pi_\mu : \mathbf{J}_Q^{-1}(\mu) \to (T^*Q)_\mu$ is the projection. Because $((T^*G)_\mu, \omega_\mu)$ is symplectically diffeomorphic to $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$, we have that $((T^*Q)_\mu, \omega_\mu)$

is symplectically diffeomorphic to $(\mathcal{O}_\mu \times V \times V^*, \tilde{\omega}^-_{\mathcal{O}_\mu \times V \times V^*})$.

We now consider the Lagrangian $L(g,\xi,\theta,\dot{\theta}) : TQ \cong G \times \mathfrak{g} \times TV \to \mathbb{R}$, which is usual the total kinetic minus potential energy of the system, where $(g,\xi) \in G \times \mathfrak{g}$, and $\theta \in V$, ξ^i and $\dot{\theta}^j = \frac{\mathrm{d}\theta^j}{\mathrm{d}t}$, $(i = 1, \cdots, n,\ j = 1, \cdots, k,\ n = \dim G,\ k = \dim V)$, regarded as the velocity of system. If we introduce the conjugate momentum $p_i = \frac{\partial L}{\partial \xi^i}$, $l_j = \frac{\partial L}{\partial \dot{\theta}^j}$, $i = 1, \cdots, n,\ j = 1, \cdots, k$, and by the Legendre transformation $FL : TQ \cong G \times \mathfrak{g} \times V \times V \to T^*Q \cong G \times \mathfrak{g}^* \times V \times V^*$, $(g^i, \xi^i, \theta^j, \dot{\theta}^j) \to (g^i, p_i, \theta^j, l_j)$, we have the Hamiltonian $H(g,p,\theta,l) : T^*Q \cong G \times \mathfrak{g}^* \times V \times V^* \to \mathbb{R}$ given by

$$H(g^i, p_i, \theta^j, l_j) = \sum_{i=1}^{n} p_i \xi^i + \sum_{j=1}^{k} l_j \dot{\theta}^j - L(g^i, \xi^i, \theta^j, \dot{\theta}^j). \tag{6.5}$$

If the Hamiltonian $H(g,p,\theta,l) : T^*Q \cong G \times \mathfrak{g}^* \times V \times V^* \to \mathbb{R}$ is left cotangent lifted G-action Φ^{T^*} invariant, for $\mu \in \mathfrak{g}^*$ we have the associated reduced Hamiltonian $h_\mu(\nu,\theta,l) : (T^*Q)_\mu \cong \mathcal{O}_\mu \times V \times V^* \to \mathbb{R}$, defined by $h_\mu \cdot \pi_\mu = H \cdot i_\mu$. Note that for $F, K : T^*V \cong V \times V^* \to \mathbb{R}$, by using the canonical symplectic form ω_V on $T^*V \cong V \times V^*$, we can define the Poisson bracket $\{\cdot,\cdot\}_V$ on T^*V as follows

$$\{F,K\}_V(\theta,\lambda) = < \frac{\delta F}{\delta \theta}, \frac{\delta K}{\delta \lambda} > - < \frac{\delta K}{\delta \theta}, \frac{\delta F}{\delta \lambda} >$$

If $\theta_i,\ i = 1, \cdots, k$, is a base of V, and $\lambda_i,\ i = 1, \cdots, k$, a base of V^*, then we have that

$$\{F,K\}_V(\theta,\lambda) = \sum_{i=1}^{k} (\frac{\partial F}{\partial \theta_i} \frac{\partial K}{\partial \lambda_i} - \frac{\partial K}{\partial \theta_i} \frac{\partial F}{\partial \lambda_i}).$$

Thus, by the $(\pm)$-Lie-Poisson brackets on $\mathfrak{g}^*$ and the Poisson bracket $\{\cdot,\cdot\}_V$ on T^*V, for $F, K : \mathfrak{g}^* \times V \times V^* \to \mathbb{R}$, we can define the Poisson bracket on $\mathfrak{g}^* \times V \times V^*$ as follows

$$\{F,K\}_\pm(\mu,\theta,\lambda) = \{F,K\}_\pm(\mu) + \{F,K\}_V(\theta,\lambda) = \pm < \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu}] > + \sum_{i=1}^{k} (\frac{\partial F}{\partial \theta_i} \frac{\partial K}{\partial \lambda_i} - \frac{\partial K}{\partial \theta_i} \frac{\partial F}{\partial \lambda_i}). \tag{6.6}$$

See Krishnaprasad and Marsden [17]. In particular, for $F_\mu, K_\mu : \mathcal{O}_\mu \times V \times V^* \to \mathbb{R}$, we have that $\tilde{\omega}^-_{\mathcal{O}_\mu \times V \times V^*}(X_{F_\mu}, X_{K_\mu}) = \{F_\mu, K_\mu\}_-|_{\mathcal{O}_\mu \times V \times V^*}$. Moreover, for reduced Hamiltonian $h_\mu(\nu,\theta,l) : \mathcal{O}_\mu \times V \times V^* \to \mathbb{R}$, we have the Hamiltonian vector field $X_{h_\mu}(K_\mu) = \{K_\mu, h_\mu\}_-|_{\mathcal{O}_\mu \times V \times V^*}$. Thus, if the Hamiltonian $H : T^*Q \to \mathbb{R}$, the fiber-preserving map $F : T^*Q \to T^*Q$ and the fiber submanifold W of T^*Q are all left cotangent lifted G-action Φ^{T^*} invariant, then we have the following theorem.

Theorem 6.3 *The 6-tuple $(T^*Q, G, \omega_0, H, F, W)$ is a regular point reducible RCH system, where $Q = G \times V$, and G is a Lie group and V is a k-dimensional vector space, and the Hamiltonian $H : T^*Q \to \mathbb{R}$, the fiber-preserving map $F : T^*Q \to T^*Q$ and the fiber submanifold W of T^*Q are all left cotangent lifted G-action Φ^{T^*} invariant. For a point $\mu \in \mathfrak{g}^*$, the regular value of the momentum map $\mathbf{J}_Q : T^*Q \to \mathfrak{g}^*$, the R_P-reduced system, that is, the 5-tuple $(\mathcal{O}_\mu \times V \times V^*, \tilde{\omega}^-_{\mathcal{O}_\mu \times V \times V^*}, h_\mu, f_\mu, W_\mu)$, is a RCH system, where $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is the coadjoint orbit, $\tilde{\omega}^-_{\mathcal{O}_\mu \times V \times V^*}$ is orbit symplectic form on $\mathcal{O}_\mu \times V \times V^*$, $h_\mu \cdot \pi_\mu = H \cdot i_\mu$, $f_\mu \cdot \pi_\mu = \pi_\mu \cdot F \cdot i_\mu$, $W \subset \mathbf{J}_Q^{-1}(\mu)$, and $W_\mu = \pi_\mu(W) \subset \mathcal{O}_\mu \times V \times V^*$. Moreover, two regular point reducible RCH system $(T^*Q_i, G_i, \omega_{i0}, H_i, F_i, W_i)$, $\ i = 1,2,$ are RpCH-equivalent if and only if the associated R_P-reduced RCH systems $(\mathcal{O}_{i\mu_i} \times V_i \times V_i^*, \tilde{\omega}^-_{\mathcal{O}_{i\mu_i}}, h_{i\mu_i}, f_{i\mu_i}, W_{i\mu_i})$, $i = 1,2,$ are RCH-equivalent.*

The third, in order to study the regular reduction of heavy top we need to the theory of Hamiltonian reduction by stages for semidirect product Lie group. See Marsden et al [21]. Assume that $S = G\circledS V$ is a semidirect product Lie group, where V is a vector space and V^* its dual space, G is a Lie group acting on the left by linear maps on V, and $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Note that G also acts on the left on the dual space V^* of V, and the action by an element g on V^* is the transpose of the action of g^{-1} on V. As a set, the underlying manifold of S is $G\times V$ and the multiplication on S is given by

$$(g_1, v_1)(g_2, v_2) := (g_1g_2, v_1 + \sigma(g_1)v_2), \quad g_1, g_1 \in G, \quad v_1, v_2 \in V \tag{6.7}$$

where $\sigma : G \to \mathrm{Aut}(V)$ is a representation of the Lie group G on V, $\mathrm{Aut}(V)$ denotes the Lie group of linear isomorphisms of V onto itself whose Lie algebra is $\mathrm{End}(V)$, the space of all linear maps of V to itself.

The Lie algebra of S is the semidirect product of Lie algebras $\mathfrak{s} = \mathfrak{g}\circledS V$, $\mathfrak{s}^*$ is the dual of $\mathfrak{s}$, that is, $\mathfrak{s}^* = (\mathfrak{g}\circledS V)^*$. The underlying vector space of $\mathfrak{s}$ is $\mathfrak{g}\times V$ and the Lie bracket on $\mathfrak{s}$ is given by

$$[(\xi_1, v_1),(\xi_2, v_2)] = ([\xi_1, \xi_2], \sigma'(\xi_1)v_2 - \sigma'(\xi_2)v_1), \quad \forall \xi_1, \xi_2 \in \mathfrak{g}, \quad v_1, v_2 \in V \tag{6.8}$$

where $\sigma' : \mathfrak{g} \to \mathrm{End}(V)$ is the induced Lie algebra representation given by

$$\sigma'(\xi)v := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sigma(\exp t\xi)v, \quad \xi \in \mathfrak{g}, \quad v \in V \tag{6.9}$$

Identify the underlying vector space of $\mathfrak{s}^*$ with $\mathfrak{g}^* \times V^*$ by using the duality pairing on each factor. One can give the formula for the $(\pm)$-Lie-Poisson bracket on the semidirect product $\mathfrak{s}^*$ as follows, that is, for $F, K : \mathfrak{s}^* \to \mathbb{R}$, their semidirect product bracket is given by

$$\{F, K\}_{\pm}(\mu, a) = \pm\langle \mu, [\frac{\delta F}{\delta\mu}, \frac{\delta K}{\delta\mu}]\rangle \pm \langle a, \frac{\delta F}{\delta\mu} \cdot \frac{\delta K}{\delta a} - \frac{\delta K}{\delta\mu} \cdot \frac{\delta F}{\delta a}\rangle \tag{6.10}$$

where $(\mu, a) \in \mathfrak{s}^*$ and $\dfrac{\delta F}{\delta\mu} \in \mathfrak{g}$, $\dfrac{\delta F}{\delta a} \in V$ are the functional derivatives. Moreover, the Hamiltonian vector field of a smooth function $H : \mathfrak{s}^* \to \mathbb{R}$ is given by

$$X_H(\mu, a) = \mp(\mathrm{ad}^*_{\delta H/\delta\mu}\, \mu - \rho^*_{\delta H/\delta a} a,\ \frac{\delta H}{\delta\mu} \cdot a), \tag{6.11}$$

where the infinitesimal action of $\mathfrak{g}$ on V can be denoted by $\xi \cdot v = \rho_v(\xi)$, for any $\xi \in \mathfrak{g}$, $v \in V$ and the map $\rho_v : \mathfrak{g} \to V$ is the derivative of the map $g \mapsto gv$ at the identity and $\rho^*_v : V^* \to \mathfrak{g}^*$ is its dual.

Now we consider a symplectic action of S on a symplectic manifold P and assume that this action has an Ad^*-equivariant momentum map $\mathbf{J}_S : P \to \mathfrak{s}^*$. On the one hand, we can regard V as a normal subgroup of S, it also acts on P and has a momentum map $\mathbf{J}_V : P \to V^*$ given by $\mathbf{J}_V = i^*_V \cdot \mathbf{J}_S$, where $i_V : V \to \mathfrak{s};\ v \mapsto (0, v)$ is the inclusion, and $i^*_V : \mathfrak{s}^* \to V^*$ is its dual. $\mathbf{J}_V$ is called the second component of $\mathbf{J}_S$. On the other hand, we can also regard G as a subgroup of S by the inclusion $i_G : G \to S$, $g \mapsto (g, 0)$. Thus, G also has a momentum map $\mathbf{J}_G : P \to \mathfrak{g}^*$ given by $\mathbf{J}_G = i^*_G \cdot \mathbf{J}_S$, which is called the first component of $\mathbf{J}_S$. Moreover, from the Ad^*-equivariance of $\mathbf{J}_S$ under G-action, we know that $\mathbf{J}_V$ is also Ad^*-equivariant under G-action. Thus, we can carry out reduction of P by S at a regular value $\sigma = (\mu, a) \in \mathfrak{s}^*$ of the momentum map $\mathbf{J}_S$

in two stages using the following procedure. (i)First reduce P by V at the value $a \in V^*$, and get the reduced space $P_a = \mathbf{J}_V^{-1}(a)/V$. Since the reduction is by the Abelian group V, so the quotient is done using the whole of V. (ii)The isometry subgroup $G_a \subset G$, consists of elements of G that leave the point $a \in V^*$ fixed using the action of G on V^*. One can prove that the group G_a leaves the set $\mathbf{J}_V^{-1}(a) \subset P$ invariant, and acts symplectically on the reduced space P_a and has a naturally induced momentum map $\mathbf{J}_a : P_a \to \mathfrak{g}_a^*$, where $\mathfrak{g}_a$ is the Lie algebra of the isometric subgroup G_a and $\mathfrak{g}_a^*$ is its dual. (iii)Reduce the first reduced space P_a at the point $\mu_a = \mu|_{\mathfrak{g}_a^*} \in \mathfrak{g}_a^*$, one can get the second reduced space $(P_a)_{\mu_a} = \mathbf{J}_a^{-1}(\mu_a)/(G_a)_{\mu_a}$. Thus, we can give the following proposition on the reduction by stages for semidirect products, see Marsden et al [21].

Proposition 6.4 *The reduced space $(P_a)_{\mu_a}$ is symplectically diffeomorphic to the reduced space P_σ obtained by reducing P by S at the regular point $\sigma = (\mu, a) \in \mathfrak{s}^*$.*

In particular, we can choose that $P = T^*S$, where $S = G \circledS V$ is a semidirect product Lie group, with the cotangent lift action of S on T^*S induced by left translations of S on itself. Since the reduction of T^*S by the action of V can give a space which is isomorphic to T^*G, from the above reduction by stages proposition for semidirect products we can get the following semidirect product reduction proposition.

Proposition 6.5 *The reduction of T^*G by G_a at the regular values $\mu_a = \mu|_{\mathfrak{g}_a^*}$ gives a space which is isomorphic to the coadjoint orbit $\mathcal{O}_\sigma \subset \mathfrak{s}^*$ through the point $\sigma = (\mu, a) \in \mathfrak{s}^*$, where $\mathfrak{s}^*$ is the dual of the Lie algebra $\mathfrak{s}$ of S.*

Thus, from the above proposition we know that the reduced space of the heavy top is obtained by the reduction of $T^*\mathrm{SE}(3)$ by left action of $\mathrm{SE}(3) = \mathrm{SO}(3) \circledS \mathbb{R}^3$, which is a coadjoint orbit in $\mathfrak{se}^*(3)$. Moreover, the configuration space of the heavy top with internal rotors is $Q = \mathrm{SE}(3) \times V$, and the reduced space is symplectically diffeomorphic to a leaf of Poisson manifold $\mathfrak{se}^*(3) \times V \times V^*$. In consequence, we can deal with uniformly the symplectic reduction of the rigid body, heavy top, as well as them with internal rotors, such that we can state that all these systems are the regular point reducible RCH systems and can give their RCH-equivalences.

6.2 Rigid Body and Heavy Top

In this subsection, by using the above method, we describe uniformly the rigid body and heavy top as well as them with internal rotors (or external force torques) as the regular point reducible RCH systems on the rotation group SO(3) and on the Euclidean group SE(3), as well as on their generalizations, respectively, and give their R_P-reduced RCH systems and discuss their RCH-equivalence. Note that our description of the motion and the equations of rigid body and heavy top follows some of the notations and conventions in Marsden and Ratiu [22], Marsden [20].

(1). Rigid Body with External Force Torque.

In the following we take Lie group $G = \mathrm{SO}(3)$, and state the rigid body with external force torque to be a regular point reducible RCH system. It is well known that, usually, the configuration space for a 3-dimensional rigid body moving freely in space is SE(3), the six dimension group of Euclidean (rigid) transformations of three dimensional space $\mathbb{R}^3$, that is, all possible rotations and translations. If translation are ignored and only rotations are considered, then the configuration space Q is SO(3), consists of all orthogonal linear transformations of

Euclidean three space to itself, which have determinant one. Its Lie algebra, denoted $\mathfrak{so}(3)$, consists of all 3×3 skew matrices, and we can identify the Lie algebra $(\mathfrak{so}(3),[,])$ with $(\mathbb{R}^3,\times)$. Denote by $\mathfrak{so}^*(3)$ the dual of the Lie algebra $\mathfrak{so}(3)$, and we also identity $\mathfrak{so}^*(3)$ with $\mathbb{R}^3$ by pairing the Euclidean inner product. Since the functional derivative of a function defined on $\mathbb{R}^3$ is equal to the usual gradient of the function, from (6.1) we know that the Lie-Poisson bracket on $\mathfrak{so}^*(3)$ take the form

$$\{f,g\}_{\pm}(\Pi)=\pm\Pi\cdot(\nabla_{\Pi}f\times\nabla_{\Pi}g),\ \ \forall f,g\in C^{\infty}(\mathfrak{so}^*(3)),\ \ \Pi\in\mathfrak{so}^*(3). \tag{6.12}$$

The phase space of a rigid body is the cotangent bundle $T^*G=T^*\mathrm{SO}(3)\cong\mathrm{SO}(3)\times\mathfrak{so}^*(3)$, with the canonical symplectic form. Assume that Lie group $G=\mathrm{SO}(3)$ acts freely and properly by the left translations on $\mathrm{SO}(3)$, then the action of $\mathrm{SO}(3)$ on the phase space $T^*\mathrm{SO}(3)$ is by cotangent lift of left translations at the identity, that is, $\Phi:\mathrm{SO}(3)\times T^*\mathrm{SO}(3)\cong\mathrm{SO}(3)\times\mathrm{SO}(3)\times\mathfrak{so}^*(3)\to\mathrm{SO}(3)\times\mathfrak{so}^*(3)$, given by $\Phi(B,(A,\Pi))=(BA,\Pi)$, for any $A,B\in\mathrm{SO}(3)$, $\Pi\in\mathfrak{so}^*(3)$, which is also free and proper, and admits an associated Ad^*-equivariant momentum map $\mathbf{J}:T^*\mathrm{SO}(3)\to\mathfrak{so}^*(3)$ for the left $\mathrm{SO}(3)$ action. If $\Pi\in\mathfrak{so}^*(3)$ is a regular value of $\mathbf{J}$, then the regular point reduced space $(T^*\mathrm{SO}(3))_{\Pi}=\mathbf{J}^{-1}(\Pi)/\mathrm{SO}(3)_{\Pi}$ is symplectically diffeomorphic to the coadjoint orbit $\mathcal{O}_{\Pi}\subset\mathfrak{so}^*(3)$.

Let I be the moment of inertia tensor computed with respect to a body fixed frame, which, in a principal body frame, we may represent by the diagonal matrix diag (I_1,I_2,I_3). Let $\Omega=(\Omega_1,\Omega_2,\Omega_3)$ be the vector of angular velocities computed with respect to the axes fixed in the body and $(\Omega_1,\Omega_2,\Omega_3)\in\mathfrak{so}(3)$. Consider the Lagrangian $L(A,\Omega):\mathrm{SO}(3)\times\mathfrak{so}(3)\to\mathbb{R}$, which is given by $L(A,\Omega)=\dfrac{1}{2}\langle\Omega,\Omega\rangle=\dfrac{1}{2}(I_1\Omega_1^2+I_2\Omega_2^2+I_3\Omega_3^2)$, where $A\in\mathrm{SO}(3)$, $(\Omega_1,\Omega_2,\Omega_3)\in\mathfrak{so}(3)$. If we introduce the conjugate angular momentum $\Pi_i=\dfrac{\partial L}{\partial\Omega_i}=I_i\Omega_i,\ i=1,2,3,$ which is also computed with respect to a body fixed frame, and by the Legendre transformation $FL:\mathrm{SO}(3)\times\mathfrak{so}(3)\to\mathrm{SO}(3)\times\mathfrak{so}^*(3),\ (A,\Omega)\to(A,\Pi)$, where $\Pi=(\Pi_1,\Pi_2,\Pi_3)\in\mathfrak{so}^*(3)$, we have the Hamiltonian $H(A,\Pi):\mathrm{SO}(3)\times\mathfrak{so}^*(3)\to\mathbb{R}$ given by

$$H(A,\Pi)=\Omega\cdot\Pi-L(A,\Omega)=\frac{1}{2}(\frac{\Pi_1^2}{I_1}+\frac{\Pi_2^2}{I_2}+\frac{\Pi_3^2}{I_3}).$$

From the above expression of the Hamiltonian, we know that $H(A,\Pi)$ is invariant under the left $\mathrm{SO}(3)$-action. For the case $\Pi_0=\mu\in\mathfrak{so}^*(3)$ is a regular value of $\mathbf{J}$, we have the reduced Hamiltonian $h_\mu(\Pi):\mathcal{O}_\mu\subset\mathfrak{so}^*(3)\to\mathbb{R}$ given by $h_\mu(\Pi)=H(A,\Pi)|_{\mathcal{O}_\mu}$. From the Lie-Poisson bracket on $\mathfrak{g}^*$, we can get the rigid body Poisson bracket on $\mathfrak{so}^*(3)$, that is, for $F,K:\mathfrak{so}^*(3)\to\mathbb{R}$, we have that $\{F,K\}_-(\Pi)=-\Pi\cdot(\nabla_\Pi F\times\nabla_\Pi K)$. In particular, for $F_\mu,K_\mu:\mathcal{O}_\mu\to\mathbb{R}$, we have that $\omega^-_{\mathcal{O}_\mu}(X_{F_\mu},X_{K_\mu})=\{F_\mu,K_\mu\}_-|_{\mathcal{O}_\mu}$. Moreover, for reduced Hamiltonian $h_\mu(\Pi):\mathcal{O}_\mu\to\mathbb{R}$, we have the Hamiltonian vector field $X_{h_\mu}(K_\mu)=\{K_\mu,h_\mu\}_-|_{\mathcal{O}_\mu}$, and hence we have that

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t}=X_{h_\mu}(\Pi)=\{\Pi,h_\mu(\Pi)\}_-|_{\mathcal{O}_\mu}=-\Pi\cdot(\nabla_\Pi\Pi\times\nabla_\Pi h_\mu)=-\nabla_\Pi\Pi\cdot(\nabla_\Pi h_\mu\times\Pi)=\Pi\times\Omega,$$

since $\nabla_\Pi\Pi=1$ and $\nabla_\Pi h_\mu=\Omega$. Thus, the equations of motion for rigid body is given by

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t}=\Pi\times\Omega. \tag{6.13}$$

From Theorem 6.2 if we consider the rigid body with a external force torque $u:T^*\mathrm{SO}(3)\to T^*\mathrm{SO}(3)$, and $u\in W\subset\mathbf{J}^{-1}(\mu)$ is invariant under the left $\mathrm{SO}(3)$-action, then the external force torque u can be regarded as a control of the rigid body, and its reduced control $u_\mu : \mathcal{O}_\mu \to \mathcal{O}_\mu$ is given by $u_\mu(\Pi) = \pi_\mu(u(A,\Pi)) = u(A,\Pi)|_{\mathcal{O}_\mu}$, where $\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathcal{O}_\mu$. Thus, the equations of motion for the rigid body with external force torques $u : T^*\mathrm{SO}(3) \to T^*\mathrm{SO}(3)$ are given by

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \Pi \times \Omega + \mathrm{vlift}(u_\mu), \tag{6.14}$$

where $\mathrm{vlift}(u_\mu) = \mathrm{vlift}(u_\mu)X_{h_\mu} \in T\mathcal{O}_\mu$. To sum up the above discussion, we have the following proposition.

Proposition 6.6 *The 5-tuple $(T^*SO(3), SO(3), \omega_0, H, u)$ is a regular point reducible RCH system. For a point $\mu \in \mathfrak{so}^*(3)$, the regular value of the momentum map $\mathbf{J} : T^*SO(3) \to \mathfrak{so}^*(3)$, the R_P-reduced system is the 4-tuple $(\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}^-, h_\mu, u_\mu)$, where $\mathcal{O}_\mu \subset \mathfrak{so}^*(3)$ is the coadjoint orbit, $\omega_{\mathcal{O}_\mu}^-$ is orbit symplectic form on $\mathcal{O}_\mu$, $h_\mu(\Pi) = H(A,\Pi)|_{\mathcal{O}_\mu}$, $u_\mu(\Pi) = \pi_\mu(u(A,\Pi)) = u(A,\Pi)|_{\mathcal{O}_\mu}$, and its equation of motion is given by (6.14).*

(2). The Rigid Body with Internal Rotors.

In the following we take Lie group $G = \mathrm{SO}(3)$, $V = S^1 \times S^1 \times S^1$, $Q = G \times V$ and state the rigid body with three symmetric internal rotors to be a regular point reducible RCH system. We consider a rigid body (to be called the carrier body) carrying three symmetric rotors. Denote by O the center of mass of the system in the body frame and at O place a set of (orthonormal) body axes. Assume that the rotor and the body coordinate axes are aligned with principal axes of the carrier body. The rotor spins under the influence of a torque u acting on the rotor. The configuration space is $Q = \mathrm{SO}(3) \times V$, where $V = S^1 \times S^1 \times S^1$, with the first factor being rigid body attitude and the second factor being the angles of rotors. The corresponding phase space is the cotangent bundle $T^*Q = T^*\mathrm{SO}(3) \times T^*V$, where $T^*V = T^*(S^1 \times S^1 \times S^1) \cong T^*\mathbb{R}^3$, with the canonical symplectic form. Assume that Lie group $G = \mathrm{SO}(3)$ acts freely and properly on Q by the left translations on $\mathrm{SO}(3)$, then the action of $\mathrm{SO}(3)$ on the phase space T^*Q is by cotangent lift of left translations on $\mathrm{SO}(3)$ at the identity, that is, $\Phi : \mathrm{SO}(3) \times T^*\mathrm{SO}(3) \times T^*V \cong \mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, given by $\Phi(B, (A, \Pi, \alpha, l)) = (BA, \Pi, \alpha, l)$, for any $A, B \in \mathrm{SO}(3)$, $\Pi \in \mathfrak{so}^*(3)$, $\alpha, l \in \mathbb{R}^3$, which is also free and proper, and admits an associated Ad^*-equivariant momentum map $\mathbf{J}_Q : T^*Q \cong \mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathfrak{so}^*(3)$ for the left $\mathrm{SO}(3)$ action. If $\Pi \in \mathfrak{so}^*(3)$ is a regular value of $\mathbf{J}_Q$, then the regular point reduced space $(T^*Q)_\Pi = \mathbf{J}_Q^{-1}(\Pi)/\mathrm{SO}(3)_\Pi$ is symplectically diffeomorphic to the coadjoint orbit $\mathcal{O}_\Pi \times \mathbb{R}^3 \times \mathbb{R}^3 \subset \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$.

Let $I = diag(I_1, I_2, I_3)$ be the moment of inertia of the carrier body in the principal body-fixed frame, and $J_i,\ i = 1,2,3$ be the moments of inertia of rotors around their rotation axes. Let $J_{ik},\ i = 1,2,3,\ k = 1,2,3$, be the moments of inertia of the ith rotor with $i = 1,2,3$, around the kth principal axis with $k = 1,2,3$, respectively, and denote by $\bar{I}_i = I_i + J_{1i} + J_{2i} + J_{3i} - J_{ii},\ i = 1,2,3$. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be the vector of body angular velocities computed with respect to the axes fixed in the body and $(\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$. Let $\alpha_i,\ i = 1,2,3$, be the relative angles of rotors and $\dot{\alpha} = (\dot{\alpha_1}, \dot{\alpha_2}, \dot{\alpha_3})$ the vector of rotor relative angular velocities about the principal axes with respect to a carrier body fixed frame. Consider the Lagrangian of the system $L(A, \Omega, \alpha, \dot{\alpha}) : \mathrm{SO}(3) \times \mathfrak{so}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, which is the total kinetic energy of the rigid body plus the total kinetic energy of rotors, given by

$$L(A, \Omega, \alpha, \dot{\alpha}) = \frac{1}{2}[\bar{I}_1\Omega_1^2 + \bar{I}_2\Omega_2^2 + \bar{I}_3\Omega_3^2 + J_1(\Omega_1 + \dot{\alpha}_1)^2 + J_2(\Omega_2 + \dot{\alpha}_2)^2 + J_3(\Omega_3 + \dot{\alpha}_3)^2],$$

where $A \in \mathrm{SO}(3)$, $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, $\dot{\alpha} = (\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3) \in \mathbb{R}^3$. If we introduce the conjugate angular momentum, given by $\Pi_i = \dfrac{\partial L}{\partial \Omega_i} = \bar{I}_i \Omega_i + J_i(\Omega_i + \dot{\alpha}_i)$, $\quad l_i = \dfrac{\partial L}{\partial \dot{\alpha}_i} = J_i(\Omega_i + \dot{\alpha}_i)$, $\quad i = 1, 2, 3$, and by the Legendre transformation $FL : \mathrm{SO}(3) \times \mathfrak{so}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, $\quad (A, \Omega, \alpha, \dot{\alpha}) \to (A, \Pi, \alpha, l)$, where $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3)$, $l = (l_1, l_2, l_3) \in \mathbb{R}^3$, we have the Hamiltonian $H(A, \Pi, \alpha, l) : \mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{aligned} H(A, \Pi, \alpha, l) &= \Omega \cdot \Pi + \dot{\alpha} \cdot l - L(A, \Omega, \alpha, \dot{\alpha}) \\ &= \frac{1}{2}[\frac{(\Pi_1 - l_1)^2}{\bar{I}_1} + \frac{(\Pi_2 - l_2)^2}{\bar{I}_2} + \frac{(\Pi_3 - l_3)^2}{\bar{I}_3} + \frac{l_1^2}{J_1} + \frac{l_2^2}{J_2} + \frac{l_3^2}{J_3}]. \end{aligned}$$

From the above expression of the Hamiltonian, we know that $H(A, \Pi, \alpha, l)$ is invariant under the left SO(3)-action. For the case $\Pi_0 = \mu \in \mathfrak{so}^*(3)$ is the regular value of $\mathbf{J}_Q$, we have the reduced Hamiltonian $h_\mu(\Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3 (\subset \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3) \to \mathbb{R}$ given by $h_\mu(\Pi, \alpha, l) = H(A, \Pi, \alpha, l)|_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$. From (6.6), the rigid body Poisson bracket on $\mathfrak{so}^*(3)$ and the Poisson bracket on $T^*\mathbb{R}^3$, we can get the Poisson bracket on T^*Q, that is, for $F, K : \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, we have that $\{F, K\}_-(\Pi, \alpha, l) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) + \{F, K\}_V(\alpha, l)$. In particular, for $F_\mu, K_\mu : \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, we have that $\tilde{\omega}^-_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}(X_{F_\mu}, X_{K_\mu}) = \{F_\mu, K_\mu\}_-|_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$. Moreover, for reduced Hamiltonian $h_\mu(\Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, we have the Hamiltonian vector field $X_{h_\mu}(K_\mu) = \{K_\mu, h_\mu\}_-|_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$, and hence we have that

$$\begin{aligned} \frac{\mathrm{d}\Pi}{\mathrm{d}t} &= X_{h_\mu}(\Pi)(\Pi, \alpha, l) = \{\Pi, h_\mu\}_-(\Pi, \alpha, l) \\ &= -\Pi \cdot (\nabla_\Pi \Pi \times \nabla_\Pi h_\mu) + \sum_{i=1}^{3} (\frac{\partial \Pi}{\partial \alpha_i}\frac{\partial h_\mu}{\partial l_i} - \frac{\partial h_\mu}{\partial \alpha_i}\frac{\partial \Pi}{\partial l_i}) = -\nabla_\Pi \Pi \cdot (\nabla_\Pi h_\mu \times \Pi) = \Pi \times \Omega, \end{aligned}$$

since $\nabla_\Pi \Pi = 1$, $\nabla_\Pi h_\mu = \Omega$ and $\frac{\partial \Pi}{\partial \alpha_i} = \frac{\partial h_\mu}{\partial \alpha_i} = 0$, $i = 1, 2, 3$. From Theorem 6.3 if we consider the rigid body-rotor system with a control torque $u : T^*Q \to T^*Q$ acting on the rotors, and $u \in W \subset \mathbf{J}_Q^{-1}(\mu)$ is invariant under the left SO(3)-action, and its reduced control torque $u_\mu : \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3$ is given by $u_\mu(\Pi, \alpha, l) = \pi_\mu(u(A, \Pi, \alpha, l)) = u(A, \Pi, \alpha, l)|_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$, where $\pi_\mu : \mathbf{J}_Q^{-1}(\mu) \to \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3$. Thus, the equations of motion for rigid body-rotor system with the control torque u acting on the rotors are given by

$$\begin{cases} \dfrac{\mathrm{d}\Pi}{\mathrm{d}t} = \Pi \times \Omega \\ \dfrac{\mathrm{d}l}{\mathrm{d}t} = \mathrm{vlift}(u_\mu) \end{cases} \tag{6.15}$$

where $\mathrm{vlift}(u_\mu) = \mathrm{vlift}(u_\mu) X_{h_\mu} \in T(\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3)$. To sum up the above discussion, we have the following proposition.

Proposition 6.7 *The 5-tuple $(T^*(SO(3) \times \mathbb{R}^3), SO(3), \omega_0, H, u)$ is a regular point reducible RCH system. For a point $\mu \in \mathfrak{so}^*(3)$, the regular value of the momentum map $\mathbf{J} : SO(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathfrak{so}^*(3)$, the R_P-reduced system is the 4-tuple $(\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3, \tilde{\omega}^-_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}, h_\mu, u_\mu)$, where $\mathcal{O}_\mu \subset \mathfrak{so}^*(3)$ is the coadjoint orbit, $\tilde{\omega}^-_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$ is orbit symplectic form on $\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3$, $h_\mu(\Pi, \alpha, l) = H(A, \Pi, \alpha, l)|_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$, $u_\mu(\Pi, \alpha, l) = \pi_\mu(u(A, \Pi, \alpha, l)) = u(A, \Pi, \alpha, l)|_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$, and its equations of motion are given by (6.15).*

(3). Heavy Top.

In the following we take Lie group $G = \mathrm{SE}(3)$ and state the heavy top to be a regular point reducible Hamiltonian system, and hence also to be a regular point reducible RCH system without the external force and control. We know that a heavy top is by definition a rigid body with a fixed point in $\mathbb{R}^3$ and moving in gravitational field. Usually, exception of the singular point, its physical phase space is $T^*\mathrm{SO}(3)$ and the symmetry group is S^1, regarded as rotations about the z-axis, the axis of gravity, this is because gravity breaks the symmetry and the system is no longer SO(3) invariant. By the semidirect product reduction theorem (See Proposition 6.5), we show that the reduction of $T^*\mathrm{SO}(3)$ by S^1 gives a space which is symplectically diffeomorphic to the reduced space obtained by the reduction of $T^*\mathrm{SE}(3)$ by left action of SE(3), that is the coadjoint orbit $\mathcal{O}_{(\mu,a)} \subset \mathfrak{se}^*(3) \cong T^*\mathrm{SE}(3)/\mathrm{SE}(3)$. In fact, in this case, we can identify the phase space $T^*\mathrm{SO}(3)$ with the reduction of the cotangent bundle of the special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \circledS \mathbb{R}^3$ by the Euclidean translation subgroup $\mathbb{R}^3$ and identifies the symmetry group S^1 with isotropy group $G_a = \{A \in \mathrm{SO}(3) \mid Aa = a\} = S^1$, which is Abelian and $(G_a)_{\mu_a} = G_a = S^1$, $\forall \mu_a \in \mathfrak{g}^*_a$, where a is a vector aligned with the direction of gravity and where SO(3) acts on $\mathbb{R}^3$ in the standard way.

Now we consider the cotangent bundle $T^*G = T^*\mathrm{SE}(3) \cong \mathrm{SE}(3) \times \mathfrak{se}^*(3)$, with the canonical symplectic form. Assume that Lie group $G = \mathrm{SE}(3)$ acts freely and properly by the left translations on SE(3), then the action of SE(3) on the phase space $T^*\mathrm{SE}(3)$ is by cotangent lift of left translations at the identity, that is, $\Phi : \mathrm{SE}(3) \times T^*\mathrm{SE}(3) \cong \mathrm{SE}(3) \times \mathrm{SE}(3) \times \mathfrak{se}^*(3) \to \mathrm{SE}(3) \times \mathfrak{se}^*(3)$, given by $\Phi((B,u),(A,v,\Pi,w)) = (BA, v, \Pi, w)$, for any $A, B \in \mathrm{SO}(3)$, $\Pi \in \mathfrak{so}^*(3)$, $u, v, w \in \mathbb{R}^3$, which is also free and proper, and admits an associated Ad^*-equivariant momentum map $\mathbf{J} : T^*\mathrm{SE}(3) \to \mathfrak{se}^*(3)$ for the left SE(3) action. If $(\Pi, w) \in \mathfrak{se}^*(3)$ is a regular value of $\mathbf{J}$, then the regular point reduced space $(T^*\mathrm{SE}(3))_{(\Pi,w)} = \mathbf{J}^{-1}(\Pi,w)/\mathrm{SE}(3)_{(\Pi,w)}$ is symplectically diffeomorphic to the coadjoint orbit $\mathcal{O}_{(\Pi,w)} \subset \mathfrak{se}^*(3)$.

Let $I = diag(I_1, I_2, I_3)$ be the moment of inertia of the heavy top in the body-fixed frame, which in principal body frame. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be the vector of heavy top angular velocities computed with respect to the axes fixed in the body and $(\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$. Let Γ be the unit vector viewed by an observer moving with the body, m be that total mass of the system, g be the magnitude of the gravitational acceleration, χ be the unit vector on the line connecting the origin O to the center of mass of the system, and h be the length of this segment. Consider the Lagrangian $L(A,v,\Omega,\Gamma) : \mathrm{SE}(3) \times \mathfrak{se}(3) \to \mathbb{R}$, which is given by $L(A,v,\Omega,\Gamma) = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) - mgh\Gamma \cdot \chi$, where $(A,v) \in \mathrm{SE}(3)$, $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$, $\Gamma \in \mathbb{R}^3$. If we introduce the conjugate angular momentum $\Pi_i = \dfrac{\partial L}{\partial \Omega_i} = I_i\Omega_i,\ i = 1,2,3$, and by the Legendre transformation $FL : \mathrm{SE}(3) \times \mathfrak{se}(3) \to \mathrm{SE}(3) \times \mathfrak{se}^*(3), \quad (A,v,\Omega,\Gamma) \to (A,v,\Pi,\Gamma)$, where $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3)$, we have the Hamiltonian $H(A,v,\Pi,\Gamma) : \mathrm{SE}(3) \times \mathfrak{se}^*(3) \to \mathbb{R}$ given by

$$H(A,v,\Pi,\Gamma) = \Omega \cdot \Pi - L(A,\Omega,,\Gamma) = \frac{1}{2}(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}) + mgh\Gamma \cdot \chi.$$

From the above expression of the Hamiltonian, we know that $H(A,v,\Pi,\Gamma)$ is invariant under the left SE(3)-action. For the case $(\Pi_0, \Gamma_0) = (\mu, a) \in \mathfrak{se}^*(3)$ is a regular value of $\mathbf{J}$, we have the reduced Hamiltonian $h_{(\mu,a)}(\Pi,,\Gamma) : \mathcal{O}_{(\mu,a)}(\subset \mathfrak{se}^*(3)) \to \mathbb{R}$ given by $h_{(\mu,a)}(\Pi,\Gamma) = H(A,v,\Pi,\Gamma)|_{\mathcal{O}_{(\mu,a)}}$. From the semidirect product bracket (6.10), we can get the heavy top

Poisson bracket on $\mathfrak{se}^*(3)$, that is, for $F, K : \mathfrak{se}^*(3) \to \mathbb{R}$, we have that

$$\{F, K\}_-(\Pi, \Gamma) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) - \Gamma \cdot (\nabla_\Pi F \times \nabla_\Gamma K - \nabla_\Pi K \times \nabla_\Gamma F).$$

In particular, for $F_{(\mu,a)}, K_{(\mu,a)} : \mathcal{O}_{(\mu,a)} \to \mathbb{R}$, we have that

$$\omega^-_{\mathcal{O}_{(\mu,a)}}(X_{F_{(\mu,a)}}, X_{K_{(\mu,a)}}) = \{F_{(\mu,a)}, K_{(\mu,a)}\}_-|_{\mathcal{O}_{(\mu,a)}}.$$

Moreover, for reduced Hamiltonian $h_{(\mu,a)}(\Pi, \Gamma) : \mathcal{O}_{(\mu,a)} \to \mathbb{R}$, we have the Hamiltonian vector field $X_{h_{(\mu,a)}}(K_{(\mu,a)}) = \{K_{(\mu,a)}, h_{(\mu,a)}\}_-|_{\mathcal{O}_{(\mu,a)}}$, and hence we have that

$$\begin{aligned}
\frac{\mathrm{d}\Pi}{\mathrm{d}t} &= X_{h_{(\mu,a)}}(\Pi) = \{\Pi, h_{(\mu,a)}(\Pi, \Gamma)\}_- \\
&= -\Pi \cdot (\nabla_\Pi \Pi \times \nabla_\Pi h_{(\mu,a)}) - \Gamma \cdot (\nabla_\Pi \Pi \times \nabla_\Gamma h_{(\mu,a)} - \nabla_\Pi h_{(\mu,a)} \times \nabla_\Gamma \Pi) \\
&= \Pi \times \Omega - mgh\chi \times \Gamma = \Pi \times \Omega + mgh\Gamma \times \chi,
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= X_{h_{(\mu,a)}}(\Gamma) = \{\Gamma, h_{(\mu,a)}(\Pi, \Gamma)\}_- \\
&= -\Pi \cdot (\nabla_\Pi \Gamma \times \nabla_\Pi h_{(\mu,a)}) - \Gamma \cdot (\nabla_\Pi \Gamma \times \nabla_\Gamma h_{(\mu,a)} - \nabla_\Pi h_{(\mu,a)} \times \nabla_\Gamma \Gamma) \\
&= \nabla_\Gamma \Gamma \cdot (\Gamma \times \nabla_\Pi h_{(\mu,a)}) = \Gamma \times \Omega,
\end{aligned}$$

since $\nabla_\Pi \Pi = 1$, $\nabla_\Gamma \Gamma = 1$, $\nabla_\Gamma \Pi = \nabla_\Pi \Gamma = 0$, and $\nabla_\Pi h_{(\mu,a)} = \Omega$. Thus, the equations of motion for heavy top is given by

$$\begin{cases} \dfrac{\mathrm{d}\Pi}{\mathrm{d}t} = \Pi \times \Omega + mgh\Gamma \times \chi, \\ \dfrac{\mathrm{d}\Gamma}{\mathrm{d}t} = \Gamma \times \Omega. \end{cases} \tag{6.16}$$

To sum up the above discussion, we have the following proposition.

Proposition 6.8 *The 4-tuple $(T^*SE(3), SE(3), \omega_0, H)$ is a regular point reducible Hamiltonian system. For a point $(\mu, a) \in \mathfrak{se}^*(3)$, the regular value of the momentum map $\mathbf{J} : T^*SE(3) \to \mathfrak{se}^*(3)$, the R_P-reduced system is the 3-tuple $(\mathcal{O}_{(\mu,a)}, \omega_{\mathcal{O}_{(\mu,a)}}, h_{(\mu,a)})$, where $\mathcal{O}_{(\mu,a)} \subset \mathfrak{se}^*(3)$ is the coadjoint orbit, $\omega_{\mathcal{O}_{(\mu,a)}}$ is orbit symplectic form on $\mathcal{O}_{(\mu,a)}$, $h_{(\mu,a)}(\Pi, \Gamma) = H(A, v, \Pi, \Gamma)|_{\mathcal{O}_{(\mu,a)}}$, and its equations of motion are given by (6.16).*

(4). The Heavy Top with Internal Rotors.

In the following we take Lie group $G = \mathrm{SE}(3)$, $V = S^1 \times S^1$, $Q = G \times V$ and state the heavy top with two pairs of symmetric internal rotors to be a regular point reducible RCH system. We shall first describe a heavy top with two pairs of symmetric rotors. We mount two pairs of rotors within the top so that each pair's rotation axis is parallel to the first and the second principal axes of the top; see Chang and Marsden [10]. The rotor spins under the influence of a torque u acting on the rotor. The configuration space is $Q = \mathrm{SE}(3) \times V$, where $V = S^1 \times S^1$, with the first factor being the position of the heavy top and the second factor being the angles of rotors. The corresponding phase space is the cotangent bundle $T^*Q = T^*\mathrm{SE}(3) \times T^*V$, where $T^*V = T^*(S^1 \times S^1) \cong T^*\mathbb{R}^2$, with the canonical symplectic form. Assume that Lie group $G = \mathrm{SE}(3)$ acts freely and properly on Q by the left translations on $\mathrm{SE}(3)$, then the action of $\mathrm{SE}(3)$ on the phase space T^*Q is by cotangent lift of left translations on $\mathrm{SE}(3)$ at the identity, that is, $\Phi : \mathrm{SE}(3) \times T^*\mathrm{SE}(3) \times T^*V \cong \mathrm{SE}(3) \times \mathrm{SE}(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathrm{SE}(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2$, given by $\Phi((B,u)((A,v),(\Pi,w),\alpha,l)) = ((BA,v),(\Pi,w),\alpha,l)$, for any $A, B \in \mathrm{SO}(3)$, $\Pi \in \mathfrak{so}^*(3)$, $u, v, w \in \mathbb{R}^3$, $\alpha, l \in \mathbb{R}^2$, which is also free and proper, and admits an associated Ad^*-equivariant momentum map $\mathbf{J}_Q : T^*Q \cong \mathrm{SE}(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathfrak{se}^*(3)$ for the left SE(3) action. If $(\Pi, w) \in \mathfrak{se}^*(3)$ is a regular value of $\mathbf{J}_Q$, then the regular point reduced space $(T^*Q)_{(\Pi,w)} = \mathbf{J}_Q^{-1}(\Pi,w)/\mathrm{SE}(3)_{(\Pi,w)}$ is symplectically diffeomorphic to the coadjoint orbit $\mathcal{O}_{(\Pi,w)} \times \mathbb{R}^2 \times \mathbb{R}^2 \subset \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2$.

Let $I = diag(I_1, I_2, I_3)$ be the moment of inertia of the heavy top in the body-fixed frame. Let $J_i, i = 1,2$ be the moments of inertia of rotors around their rotation axes. Let $J_{ik},\ i = 1,2,\ k = 1,2,3$, be the moments of inertia of the i-th rotor with $i = 1,2$ around the k-th principal axis with $k = 1,2,3$, respectively, and denote by $\bar{I}_i = I_i + J_{1i} + J_{2i} - J_{ii},\ i = 1,2$, and $\bar{I}_3 = I_3 + J_{13} + J_{23}$. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be the vector of heavy top angular velocities computed with respect to the axes fixed in the body and $(\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$. Let $\theta_i,\ i = 1,2$, be the relative angles of rotors and $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2)$ the vector of rotor relative angular velocities about the principal axes with respect to the body fixed frame of heavy top. Let m be that total mass of the system, g be the magnitude of the gravitational acceleration and h be the distance from the origin O to the center of mass of the system. Consider the Lagrangian $L(A, v, \Omega, \Gamma, \theta, \dot{\theta}) : \mathrm{SE}(3) \times \mathfrak{se}(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, which is the total kinetic energy of the heavy top plus the total kinetic energy of rotors minus potential energy of the system, given by

$$L(A, v, \Omega, \Gamma, \theta, \dot{\theta}) = \frac{1}{2}[\bar{I}_1\Omega_1^2 + \bar{I}_2\Omega_2^2 + \bar{I}_3\Omega_3^2 + J_1(\Omega_1 + \dot{\theta}_1)^2 + J_2(\Omega_2 + \dot{\theta}_2)^2] - mgh\Gamma \cdot \chi,$$

where $(A, v) \in \mathrm{SE}(3)$, $(\Omega, \Gamma) \in \mathfrak{se}(3)$ and $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$, $\Gamma \in \mathbb{R}^3$, $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$, $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2) \in \mathbb{R}^2$. If we introduce the conjugate angular momentum, which is given by $\Pi_i = \dfrac{\partial L}{\partial \Omega_i} = \bar{I}_i\Omega_i + J_i(\Omega_i + \dot{\theta}_i),\ i = 1,2,\quad \Pi_3 = \dfrac{\partial L}{\partial \Omega_3} = \bar{I}_3\Omega_3,\quad l_i = \dfrac{\partial L}{\partial \dot{\theta}_i} = J_i(\Omega_i + \dot{\theta}_i),\ i = 1,2$, and by the Legendre transformation $FL : \mathrm{SE}(3) \times \mathfrak{se}(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathrm{SE}(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2,\quad (A, v, \Omega, \Gamma, \theta, \dot{\theta}) \to (A, v, \Pi, \Gamma, \theta, l)$, where $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3)$, $l = (l_1, l_2) \in \mathbb{R}^2$, we have the Hamiltonian $H(A, v, \Pi, \Gamma, \theta, l) : \mathrm{SE}(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} H(A, v, \Pi, \Gamma, \theta, l) &= \Omega \cdot \Pi + \dot{\theta} \cdot l - L(A, v, \Omega, \Gamma, \theta, \dot{\theta}) \\ &= \frac{1}{2}[\frac{(\Pi_1 - l_1)^2}{\bar{I}_1} + \frac{(\Pi_2 - l_2)^2}{\bar{I}_2} + \frac{\Pi_3^2}{\bar{I}_3} + \frac{l_1^2}{J_1} + \frac{l_2^2}{J_2}] + mgh\Gamma \cdot \chi. \end{aligned}$$

From the above expression of the Hamiltonian, we know that $H(A, v, \Pi, \Gamma, \theta, l)$ is invariant under the left SE(3)-action. For the case $(\Pi_0, \Gamma_0) = (\mu, a) \in \mathfrak{se}^*(3)$ is the regular value of $\mathbf{J}_Q$, we have the reduced Hamiltonian $h_{(\mu,a)}(\Pi, \Gamma, \theta, l) : \mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2 (\subset \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}$ given by $h_{(\mu,a)}(\Pi, \Gamma, \theta, l) = H(A, v, \Pi, \Gamma, \theta, l)|_{\mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2}$. From (6.6), the heavy top Poisson bracket on $\mathfrak{se}^*(3)$ and the Poisson bracket on $T^*\mathbb{R}^2$, we can get the Poisson bracket on T^*Q, that is, for $F, K : \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, we have that

$$\{F, K\}_-(\Pi, \Gamma, \theta, l) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) - \Gamma \cdot (\nabla_\Pi F \times \nabla_\Gamma K - \nabla_\Pi K \times \nabla_\Gamma F) + \{F, K\}_V(\theta, l).$$

In particular, for $F_{(\mu,a)}, K_{(\mu,a)} : \mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, we have that

$$\tilde{\omega}^-_{\mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2}(X_{F_{(\mu,a)}}, X_{K_{(\mu,a)}}) = \{F_{(\mu,a)}, K_{(\mu,a)}\}_-|_{\mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2}.$$

Moreover, for reduced Hamiltonian $h_{(\mu,a)}(\Pi,\Gamma): \mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$, we have the Hamiltonian vector field $X_{h_{(\mu,a)}}(K_{(\mu,a)}) = \{K_{(\mu,a)}, h_{(\mu,a)}\}_-|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2}$, and hence we have that

$$\begin{aligned}
\frac{\mathrm{d}\Pi}{\mathrm{d}t} &= X_{h_{(\mu,a)}}(\Pi)(\Pi,\Gamma,\theta,l) = \{\Pi, h_{(\mu,a)}\}_-(\Pi,\Gamma,\theta,l) = -\Pi\cdot(\nabla_\Pi\Pi\times\nabla_\Pi h_{(\mu,a)}) \\
&\quad - \Gamma\cdot(\nabla_\Pi\Pi\times\nabla_\Gamma h_{(\mu,a)} - \nabla_\Pi h_{(\mu,a)}\times\nabla_\Gamma\Pi) + \sum_{i=1}^{2}(\frac{\partial\Pi}{\partial\theta_i}\frac{\partial h_{(\mu,a)}}{\partial l_i} - \frac{\partial h_{(\mu,a)}}{\partial\theta_i}\frac{\partial\Pi}{\partial l_i}) \\
&= \Pi\times\Omega - mgh\chi\times\Gamma = \Pi\times\Omega + mgh\Gamma\times\chi,
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= X_{h_{(\mu,a)}}(\Gamma)(\Pi,\Gamma,\theta,l) = \{\Gamma, h_{(\mu,a)}\}_-(\Pi,\Gamma,\theta,l) = -\Pi\cdot(\nabla_\Pi\Gamma\times\nabla_\Pi h_{(\mu,a)}) \\
&\quad - \Gamma\cdot(\nabla_\Pi\Gamma\times\nabla_\Gamma h_{(\mu,a)} - \nabla_\Pi h_{(\mu,a)}\times\nabla_\Gamma\Gamma) + \sum_{i=1}^{2}(\frac{\partial\Gamma}{\partial\theta_i}\frac{\partial h_{(\mu,a)}}{\partial l_i} - \frac{\partial h_{(\mu,a)}}{\partial\theta_i}\frac{\partial\Gamma}{\partial l_i}) \\
&= \nabla_\Gamma\Gamma\cdot(\Gamma\times\nabla_\Pi h_{(\mu,a)}) = \Gamma\times\Omega,
\end{aligned}$$

since $\nabla_\Pi\Pi = 1,\ \nabla_\Gamma\Gamma = 1,\ \nabla_\Gamma\Pi = \nabla_\Pi\Gamma = 0,\ \nabla_\Pi h_{(\mu,a)} = \Omega$, and $\frac{\partial\Pi}{\partial\theta_i} = \frac{\partial\Gamma}{\partial\theta_i} = \frac{\partial h_{(\mu,a)}}{\partial\theta_i} = 0,\ i=1,2$. From Theorem 6.3 if we consider the heavy top-rotor system with a control torque $u: T^*Q \to T^*Q$ acting on the rotors, and $u \in W \subset \mathbf{J}_Q^{-1}((\mu,a))$ is invariant under the left SE(3)-action, and its reduced control torque $u_{(\mu,a)}: \mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2$ is given by $u_{(\mu,a)}(\Pi,\Gamma,\theta,l) = \pi_{(\mu,a)}(u(A,v,\Pi,\Gamma,\theta,l)) = u(A,v,\Pi,\Gamma,\theta,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2}$, where $\pi_{(\mu,a)}: \mathbf{J}_Q^{-1}((\mu,a)) \to \mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2$. Thus, the equations of motion for heavy top-rotor system with the control torque u acting on the rotors are given by

$$\begin{cases}
\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \Pi\times\Omega + mgh\Gamma\times\chi, \\
\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \Gamma\times\Omega, \\
\frac{\mathrm{d}l}{\mathrm{d}t} = \mathrm{vlift}(u_{(\mu,a)}).
\end{cases} \tag{6.17}$$

where $\mathrm{vlift}(u_{(\mu,a)}) = \mathrm{vlift}(u_{(\mu,a)})X_{h_{(\mu,a)}} \in T(\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2)$. To sum up the above discussion, we have the following proposition.

Proposition 6.9 *The 5-tuple $(T^*(SE(3)\times\mathbb{R}^2), SE(3), \omega_0, H, u)$ is a regular point reducible RCH system. For a point $(\mu,a)\in\mathfrak{se}^*(3)$, the regular value of the momentum map $\mathbf{J}: SE(3)\times\mathfrak{se}^*(3)\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathfrak{se}^*(3)$, the R_P-reduced system is the 4-tuple $(\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2,\ \tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2},\ h_{(\mu,a)},\ u_{(\mu,a)})$, where $\mathcal{O}_{(\mu,a)} \subset \mathfrak{se}^*(3)$ is the coadjoint orbit, $\tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2}$ is orbit symplectic form on $\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2$, $h_{(\mu,a)}(\Pi,\Gamma,\theta,l) = H(A,v,\Pi,\Gamma,\theta,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2}$, and $u_{(\mu,a)}(\Pi,\Gamma,\theta,l) = \pi_{(\mu,a)}(u(A,v,\Pi,\Gamma,\theta,l)) = u(A,v,\Pi,\Gamma,\theta,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}^2\times\mathbb{R}^2}$, and its equations of motion are given by (6.17).*

(5). Regular Controlled Hamiltonian Equivalence.

In the following we shall state the RCH-equivalences of the rigid body with external force torques and that with internal rotors, as well as the heavy top and that with internal rotors. In fact, we can choose the feedback control law such that the equivalent RCH systems produce the same equations of motion (up to a diffeomorphism).

At first, we consider the RCH-equivalence between the rigid body with external force torques and that with internal rotors. Now let us choose the feedback control laws such that the closed-loop systems are Hamiltonian and retain the symmetry. If we choose the feedback control law u, such that $\mathrm{vlift}(u_\mu) = p \times \Omega$, where p is a constant vector, from the equations (6.14) of motion for the rigid body with the SO(3)-invariant external force torque u, we have that

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \Pi \times \Omega + p \times \Omega. \tag{6.18}$$

On the other hand, for the rigid body with internal rotors, we choose the feedback control law u, such that $\mathrm{vlift}(u_\mu) = k(\Pi \times \Omega)$, where k is a gain parameter. From the equations (6.15) of motion for the rigid body with internal rotors, we have that $\frac{\mathrm{d}l}{\mathrm{d}t} = \mathrm{vlift}(u_\mu) = k\frac{\mathrm{d}\Pi}{\mathrm{d}t}$, and by solving the integrable equation, we get that $l - k\Pi = p$, where p is a constant vector. Assuming that $N = \Pi - l = \Pi - k\Pi - p = (1-k)\Pi - p$, then we have that

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}\Pi}{\mathrm{d}t} - \frac{\mathrm{d}l}{\mathrm{d}t} = (1-k)\Pi \times \Omega = N \times \Omega + p \times \Omega. \tag{6.19}$$

By comparing (6.18) and (6.19) we know that the rigid body with external force torque and that with internal rotors are RCH-equivalent by a diffeomorphism $\varphi : \mathfrak{so}^*(3) \to \mathfrak{so}^*(3), \Pi \to N$. In particular, if we take that $\mathrm{vlift}(u_\mu) = (u_{\mu 1}, u_{\mu 2}, u_{\mu 3}) = (0, 0, -\varepsilon\frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2) \in \mathbb{R}^3$, we recover the result in Bloch et al. [6], also see Marsden [20].

Next, we consider the RCH-equivalence between the rigid body with internal rotors and heavy top. If assuming that $N = \Pi + \Gamma$, from the equations (6.16) of motion for the heavy top, we have that

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N \times \Omega + mgh\Gamma \times \chi = N \times \Omega - mgh\chi \times \Gamma$$

Thus, take that $\Gamma = \lambda\Omega$ and $p = -mgh\lambda\chi$, where λ is a constant, then

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N \times \Omega + p \times \Omega. \tag{6.20}$$

In this case, by comparing (6.19) and (6.20) we know that the heavy top and the rigid body with internal rotors are RCH-equivalent. In the same way, from (6.18) we know that the rigid body with the external force torques and the heavy top are also RCH-equivalent. Also see Holm and Marsden [15].

At last, we consider the RCH-equivalence between the rigid body with internal rotors and heavy top with internal rotors. For the heavy top with internal rotors, we choose the feedback control law u, such that $\mathrm{vlift}(u_{(\mu,a)}) = k(\Gamma \times \Omega)$, where k is a gain parameter. From the equations (6.17) of motion for the heavy top with internal rotors, we have that $\frac{\mathrm{d}\bar{l}}{\mathrm{d}t} = \mathrm{vlift}(u_{(\mu,a)}) = k\frac{\mathrm{d}\Gamma}{\mathrm{d}t}$, where $\bar{l} = (l_1, l_2, 0)$, and by solving the integrable equation, we get that $\bar{l} - k\Gamma = p_0$, where p_0 is a constant vector. Assuming that $N = \Pi + \Gamma - \bar{l} = \Pi + (1-k)\Gamma - p_0$, then we have that

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}\Pi}{\mathrm{d}t} + \frac{\mathrm{d}\Gamma}{\mathrm{d}t} - \frac{\mathrm{d}\bar{l}}{\mathrm{d}t} = \Pi \times \Omega + (1-k)\Gamma \times \Omega - mgh\chi \times \Gamma = N \times \Omega + p_0 \times \Omega - mgh\chi \times \Gamma.$$

Thus, take that $\Gamma = \lambda\Omega$ and $p = p_0 - mgh\lambda\chi$, where λ is a constant, then

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N \times \Omega + p \times \Omega. \tag{6.21}$$

In this case, by comparing (6.19) and (6.21) we know that the rigid body with internal rotors and the heavy top with internal rotors are RCH-equivalent.

To sum up, we have the following theorem.

Theorem 6.10 *As two R_P-reduced RCH systems,*
(**i**) *the rigid body with external force torque and that with internal rotors are RCH-equivalent;*
(**ii**) *the rigid body with internal rotors (or external force torque) and the heavy top are RCH-equivalent;*
(**iii**) *the rigid body with internal rotors and the heavy top with internal rotors are RCH-equivalent.*

6.3 Port Hamiltonian System with a Symplectic Structure

In order to understand well the abstract definition of RCH system and the RCH-equivalence, in this subsection we will describe the RCH system and RCH-equivalence from the viewpoint of port Hamiltonian system with a symplectic structure. Recently years, the study of stability analysis and control of port Hamiltonian systems and their applications have become more and more important, and there have been a lot of beautiful results; see Dalsmo and van der Schaft [13], van der Schaft [30, 31]. To describe the RCH systems well from the viewpoint of port Hamiltonian system, in the following we first give some relevant definitions and basic facts about the port Hamiltonian systems.

Definition 6.11 *Let (T^*Q, ω) be a symplectic manifold and ω be the canonical symplectic form on T^*Q. Assume that $H : T^*Q \to \mathbb{R}$ is a Hamiltonian, and there exists a subset $U \subset T^*Q$ and a vector field $X_H \in TT^*Q$ on T^*Q such that $i_{X_H}\omega(z) = \mathbf{d}H(z)$, $\forall z \in U$, then the triple (T^*Q, ω, H) is a Hamiltonian system defined on the set U. Assume that $V \subset T^*Q$ is a subset of T^*Q, and $P = (Y, \alpha)$, where for any $z \in V$, $Y(z) \in T_zT^*Q$ and $\alpha(z) \in T_z^*T^*Q$. If $U \cap V \neq \emptyset$, and $i_{(X_H+Y)}\omega(z) = (\mathbf{d}H + \alpha)(z)$, $\forall z \in U \cap V$, then $P = (Y, \alpha)$ is called a port of the Hamiltonian system (T^*Q, ω, H) defined on the set U. The 4-tuple (T^*Q, ω, H, P) is called a port Hamiltonian system.*

For the port Hamiltonian system (T^*Q, ω, H, P), since $i_{X_H}\omega(z) = \mathbf{d}H(z)$, $\forall z \in U$, from $i_{(X_H+Y)}\omega(z)$
$= (\mathbf{d}H + \alpha)(z)$, $\forall z \in U \cap V$, we have that $i_{X_H}\omega(z) + i_Y\omega(z) = \mathbf{d}H(z) + \alpha(z)$. Thus, we can get the port balance condition that $P = (Y, \alpha)$ is a port of the Hamiltonian system (T^*Q, ω, H) as follows

$$i_Y\omega(z) = \alpha(z), \quad \forall z \in U \cap V. \tag{6.22}$$

In particular, for $U = V = T^*Q$, from the port balance condition (6.22) we know that $P = (X_H, \mathbf{d}H)$ is a trivial port of the Hamiltonian system (T^*Q, ω, H).

Assume that (T^*Q, ω, H, F, u) is a RCH system with a control law u. We can take that $Y = \mathrm{vlift}(F + u) \in TT^*Q$, from the port balance condition (6.22) we take that $\alpha = i_Y\omega \in T^*T^*Q$, then $P = (Y, \alpha)$ is a force-controlled port of the Hamiltonian system (T^*Q, ω, H), and (T^*Q, ω, H, P) is a port Hamiltonian system with a symplectic structure. Thus, we have the following proposition.

Proposition 6.12 *Any RCH system (T^*Q, ω, H, F, u) with control law u, is a port Hamiltonian system with symplectic structure.*

If we consider the canonical coordinates $z = (q, p)$ of the phase space T^*Q, then $X_H = (\dot{q}, \dot{p})$, and the local expression of the RCH system is given by

$$\dot{q} = \frac{\partial H}{\partial p}(q, p), \qquad \dot{p} = -\frac{\partial H}{\partial q}(q, p) + \text{vlift}(F + u)(q, p). \tag{6.23}$$

We can derive the energy balance condition, that is,

$$\frac{dH}{dt} = (\frac{\partial H}{\partial q})^T(q, p)\dot{q} + (\frac{\partial H}{\partial p})^T(q, p)\dot{p} = (\frac{\partial H}{\partial p})^T\text{vlift}(F + u)(q, p) = \dot{q}^T\text{vlift}(F + u)(q, p), \tag{6.24}$$

which expresses that the increase in energy of the system is equal to the supplied work (that is, conservation of energy). This motivates to define the output of the system as $e = \dot{q}$, which is considered as the vector of generalized velocities, and the local expression of the port controlled Hamiltonian system is given by

$$\dot{q} = \frac{\partial H}{\partial p}(q, p), \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p) + B(q)f, \quad e = B^T(q)\dot{q}. \tag{6.25}$$

where $\text{vlift}(F + u) = B(q)f$, and f is a input of system; see van der Schaft [30,31].

In the following we shall state the relationships between RCH-equivalence of RCH systems and the equivalence of port Hamiltonian systems. We first give the definitions of equivalence of Hamiltonian systems, port-equivalence of port Hamiltonian systems and equivalence of port Hamiltonian systems as follows. Assume that (T^*Q_i, ω_i), $i = 1, 2$, are two symplectic manifolds, and $\psi : T^*Q_1 \to T^*Q_2$ is a symplectic diffeomorphism. Let $T\psi : TT^*Q_1 \to TT^*Q_2$ be the tangent map of $\psi : T^*Q_1 \to T^*Q_2$, and $\psi_* = (\psi^{-1})^* : T^*T^*Q_1 \to T^*T^*Q_2$ be the cotangent map of $\psi^{-1} : T^*Q_2 \to T^*Q_1$. Then we can describe the equivalence of the Hamiltonian systems as follows.

Definition 6.13 *Assume that (T^*Q_i, ω_i, H_i), $i = 1, 2$, are two Hamiltonian systems. We say them to be equivalent, if there exists a symplectic diffeomorphism $\psi : T^*Q_1 \to T^*Q_2$, such that $T\psi(X_{H_1}) = X_{H_2} \cdot \psi$, $\psi_*(\mathbf{d}H_1) = \mathbf{d}H_2 \cdot \psi$, where $i_{X_{H_i}}\omega = \mathbf{d}H_i$, $i = 1, 2$.*

Moreover, we can describe the port-equivalence of port Hamiltonian systems and the equivalence of port Hamiltonian systems as follows.

Definition 6.14 *Assume that $(T^*Q_i, \omega_i, H_i, P_i)$, $i = 1, 2$, are two port Hamiltonian systems. We say them to be port -equivalent, if there exists a diffeomorphism $\psi : T^*Q_1 \to T^*Q_2$, such that $T\psi(Y_1) = Y_2 \cdot \psi$, $\psi_*(\alpha_1) = \alpha_2 \cdot \psi$, where $P_i = (Y_i, \alpha_i)$, and for any $z_i \in V_i(\subset T^*Q_i)$, $Y_i(z_i) \in T_{z_i}T^*Q_i$ and $\alpha_i(z_i) \in T^*_{z_i}T^*Q_i$, $i = 1, 2$. Furthermore, we say two port Hamiltonian systems $(T^*Q_i, \omega_i, H_i, P_i)$, $i = 1, 2$, to be equivalent, if there exists a diffeomorphism $\psi : T^*Q_1 \to T^*Q_2$, such that not only two Hamiltonian systems (T^*Q_i, ω_i, H_i), $i = 1, 2$, are equivalent, but also their ports are equivalent.*

Thus, we can obtain the following theorem.

Theorem 6.15 *(**i**) If two RCH systems $(T^*Q_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, are RCH-equivalent and their associated Hamiltonian systems (T^*Q_i, ω_i, H_i), $i = 1, 2$, are also equivalent, then they must be equivalent for port Hamiltonian systems.*
*(**ii**) If two RCH systems $(T^*Q_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, are RCH-equivalent, but the associated Hamiltonian systems (T^*Q_i, ω_i, H_i), $i = 1, 2$, are not equivalent, then we can choose the control law u_i, such that they are port-equivalent for port Hamiltonian systems.*

Proof. (**i**) In fact, assume that two RCH systems $(T^*Q_i,\omega_i,H_i,F_i,W_i),\ i=1,2$, are RCH-equivalent, then there exists a diffeomorphism $\varphi: Q_1\to Q_2$, such that $\varphi^*: T^*Q_2\to T^*Q_1$ is symplectic, and from Theorem 3.3 there exist two control laws $u_i: T^*Q_i\to W_i,\ i=1,2$, such that the two associated closed-loop systems produce the same equations of motion, that is, $X_{(T^*Q_1,\omega_1,H_1,F_1,u_1)}\cdot\varphi^* = T\varphi^* X_{(T^*Q_2,\omega_2,H_2,F_2,u_2)}$. If the associated Hamiltonian systems (T^*Q_i,ω_i,H_i), $i=1,2$ are also equivalent, from $\varphi_*=(\varphi^{-1})^*: T^*Q_1\to T^*Q_2$ is symplectic, and $T\varphi_*(X_{H_1})=X_{H_2}\cdot\varphi_*$, and $X_{H_i}=(\mathbf{d}H_i)^\sharp,\ i=1,2$, we have that $T\varphi^*(\mathbf{d}H_2)^\sharp=(\mathbf{d}H_1)^\sharp\cdot\varphi^*$. Note that $X_{(T^*Q_i,\omega_i,H_i,F_i,u_i)}=(\mathbf{d}H_i)^\sharp+\mathrm{vlift}(F_i)+\mathrm{vlift}(u_i),\ i=1,2$, then, $T\varphi^*(\mathrm{vlift}(F_2)+\mathrm{vlift}(u_2))=(\mathrm{vlift}(F_1)+\mathrm{vlift}(u_1))\cdot\varphi^*$. We can first take that $Y_i=\mathrm{vlift}(F_i+u_i)\in TT^*Q_i,\ i=1,2$, then we have that $T\varphi^*(Y_2)=Y_1\cdot\varphi^*$, and hence $T\varphi_*(Y_1)=Y_2\cdot\varphi_*$. Then we take that $\alpha_i=i_{Y_i}\omega_i\in T^*T^*Q_i,\ i=1,2$. Since the map $(\varphi_*)_*=(\varphi_*^{-1})^*: T^*T^*Q_1\to T^*T^*Q_2$, such that $(\varphi_*)_*(i_{Y_1}\omega_1)=i_{T\varphi_*(Y_1)}(\varphi_*)_*(\omega_1)=i_{Y_2}\omega_2\cdot\varphi_*$, we have that $(\varphi_*)_*(\alpha_1)=\alpha_2\cdot\varphi_*$. Thus, the ports $P_i=(Y_i,\alpha_i)$, satisfying $T\varphi_*(Y_1)=Y_2\cdot\varphi_*$, and $(\varphi_*)_*(\alpha_1)=\alpha_2\cdot\varphi_*$, are equivalent, and hence the port Hamiltonian systems $(T^*Q_i,\omega_i,H_i,P_i),\ i=1,2$, are equivalent.

(**ii**) Assume that two RCH systems $(T^*Q_i,\omega_i,H_i,F_i,W_i),\ i=1,2$, are RCH-equivalent, but the associated Hamiltonian systems $(T^*Q_i,\omega_i,H_i),\ i=1,2$, are not equivalent, from Theorem 3.3 we can choose the control law $u_i: T^*Q_i\to W_i,\ i=1,2$, such that $T(\varphi^*)\cdot X_{(T^*Q_2,\omega_2,H_2,F_2,u_2)}=X_{(T^*Q_1,\omega_1,H_1,F_1,u_1)}\cdot\varphi^*$, and hence $T(\varphi_*)\cdot X_{(T^*Q_1,\omega_1,H_1,F_1,u_1)}=X_{(T^*Q_2,\omega_2,H_2,F_2,u_2)}\cdot\varphi_*$. We can take that $Y_i=X_{(T^*Q_i,\omega_i,H_i,F_i,u_i)}=(\mathbf{d}H_i)^\sharp+\mathrm{vlift}(F_i)+\mathrm{vlift}(u_i)\in TT^*Q_i$, and $\alpha_i=i_{Y_i}\omega_i\in T^*T^*Q_i,\ i=1,2$. Then the ports $P_i=(Y_i,\alpha_i),\ i=1,2$, satisfy that $T\varphi_*(Y_1)=Y_2\cdot\varphi_*$, and $(\varphi_*)_*(\alpha_1)=\alpha_2\cdot\varphi_*$, and hence the port Hamiltonian systems $(T^*Q_i,\omega_i,H_i,P_i),\quad i=1,2$, are port-equivalent. ■

The theory of mechanical control system is a very important subject. In this paper, we study the regular reduction theory of controlled Hamiltonian systems with the symplectic structure and symmetry. It is a natural problem what and how we could do, if we define a controlled Hamiltonian system on the cotangent bundle T^*Q by using a Poisson structure, and if symplectic reduction procedure does not work or is not efficient enough. Wang and Zhang in [32] study the optimal reduction theory of controlled Hamiltonian systems with Poisson structure and symmetry by using the optimal momentum map.

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